The Alternating Group Generated by 3-Cycles

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Framework

Generated group: a pair (G, A) with G a group and $A \subseteq G$ a set generating G as a monoid.
Assume in addition that A is closed under G-conjugation. Let $e \in G$ be the identity.

Let $[n] = \{1, 2, \ldots, n\}$.

The Alternating Group Generated by 3-Cycles

Let $G = \mathfrak{A}_N = \{g \in \mathfrak{S}_N \mid \mathrm{sgn}(g) = 1\}$, and $A = \{(i\,j\,k), (i\,k\,j) \mid 1 \leq i < j < k \leq N\}$.

Write ℓ_3 instead of ℓ_A, and Red_3 instead of Red_A.

The Symmetric Group Generated by 2-Cycles

Let $G = \mathfrak{S}_N = \{g : [N] \to [N] \mid g \text{ bijective}\}$ and $A = \{(i\,j) \mid 1 \leq i < j \leq N\}$.

Write ℓ_2 instead of ℓ_A, and Red_2 instead of Red_A.

Length Function and Prefix Order

Let $g, h \in G$.

***A*-Length**: $\ell_A(g) = \min\{k \mid g = a_1 a_2 \cdots a_k, a_i \in A\}$.
***A*-Prefix order**: $g \leq_A h$ if $\ell_A(h) = \ell_A(g) + \ell_A(g^{-1}h)$.

Let $g \in \mathfrak{A}_N$ and let $\mathrm{ocyc}(g)$ denote the number of odd cycles of g.

Proposition 1 (Mühle & Nadeau, 2017) *We have*

$$\ell_3(g) = \frac{N - ocyc(g)}{2}.$$

Also: Herzog & Reid, 1976

Let $g \in \mathfrak{S}_N$ and let $\mathrm{cyc}(g)$ denote the number of cycles of g.

Proposition 2 (Folklore) *We have*

$$\ell_2(g) = N - cyc(g).$$

Interval Structure

For $g \in G$ study the interval $[e, g]_A$ in $(G, \leq_A)$.

Let $g = \xi\zeta_1\zeta_2 \cdots \zeta_r \in \mathfrak{A}_N$, where each ζ_i is an odd cycle, and ξ is a product of even cycles.

Proposition 3 (Mühle & Nadeau, 2017) *We have*

$$[e, g]_3 = [e, \xi]_3 \times \prod_{i=1}^{r} [e, \zeta_i]_3.$$

Let $g = \zeta_1\zeta_2 \cdots \zeta_r \in \mathfrak{S}_N$, where each ζ_i is a cycle.

Proposition 4 (Biane, 1997) *We have*

$$[e, g]_2 = \prod_{i=1}^{r} [e, \zeta_i]_2.$$

Enumeration

Fix $g \in G$.

***m*-multichain**: m-tuple $(g_1, g_2, \ldots, g_m)$ with $g_1 \leq_A g_2 \leq_A \cdots \leq_A g_m \leq_A g$.
Zeta polynomial: $\mathcal{Z}_g(m)$ counts $m-1$-multichains.
Rank jump enumeration: $\mathcal{R}_g(m; r_1, r_2, \ldots, r_{m+1})$ counts m-multichains with $r_i = \ell_A(g_i) - \ell_A(g_{i-1})$, where $g_0 = e$ and $g_{m+1} = g$.

Chain enumeration in $[e, g]_A$.

Let $N = 2n + 1$ and suppose that g is an N-cycle.

Theorem 5 (Mühle & Nadeau, 2017) *We have*

$$\mathcal{Z}_g(m) = \frac{m}{mN - n}\binom{mN - n}{n}.$$

Theorem 6 (Mühle & Nadeau, 2017) *We have*

$$\mathcal{R}_g(m; r_1, r_2, \ldots, r_{m+1}) = \frac{1}{N}\prod_{i=1}^{m+1} \frac{N}{N - r_i}\binom{N - r_i}{r_i}.$$

This extends to groups generated by k-cycles.

Let $N = n + 1$, and suppose that g is an N-cycle.

Theorem 7 (Kreweras, 1972) *We have*

$$\mathcal{Z}_g(m) = \frac{1}{N}\binom{mN}{n}.$$

Theorem 8 (Edelman, 1980) *We have*

$$\mathcal{R}_g(m; r_1, r_2, \ldots, r_{m+1}) = \frac{1}{N}\prod_{i=1}^{m+1}\binom{N}{r_i}.$$

Noncrossing partition lattice!

Hurwitz Orbits

Braid generator: σ_i exchanges i^{th} and $(i+1)^{\text{st}}$ strand.
Braid group: group $\mathfrak{B}_n$ generated by $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ subject to braid relations.

Assume that $\ell_A(g) = k$.
***A*-reduced factorization**: any product $g = a_1 a_2 \cdots a_k$.
Let $\mathrm{Red}_A(g)$ denote the set of all A-reduced factorizations of g.

Hurwitz move: for $j < n$ define
$\sigma_j \cdot (a_1 \cdots a_j a_{j+1} \cdots a_k) = a_1 \cdots a_{j+1}(a_{j+1}^{-1} a_j a_{j+1}) \cdots a_k$.
Hurwitz action: action of $\mathfrak{B}_k$ on $\mathrm{Red}_A(g)$ defined by Hurwitz moves.

Theorem 9 (Mühle & Nadeau, 2017) *Let $g \in \mathfrak{A}_N$ have $2k$ even cycles. The Hurwitz action on $\mathrm{Red}_3(g)$ has $\frac{(2k)!}{k!}$ orbits.*

Sketch of proof

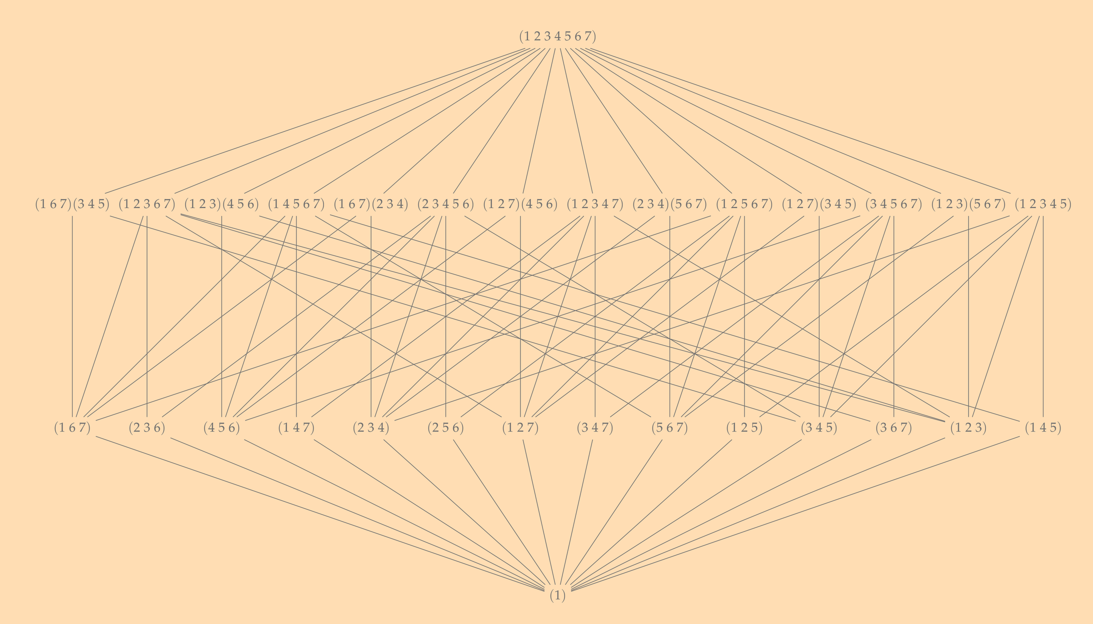
- prove Hurwitz transitivity for $k = 0$
- for $k = 1$, partition the generators into mixed and pure
- define a parity function on mixed generators
- show that parity is preserved under Hurwitz action
- for $k > 1$, any matching of the even cycles of g is invariant under Hurwitz action

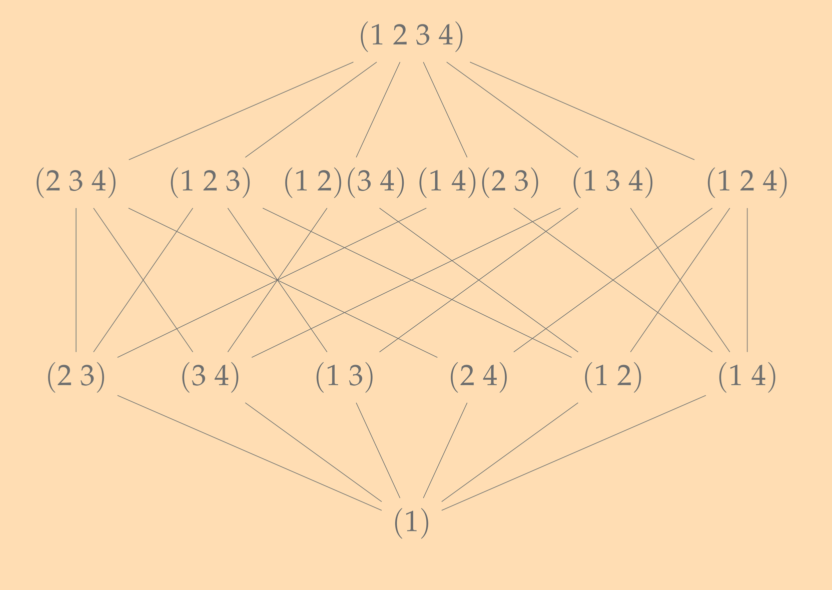
Theorem 10 (Deligne, 1974) *For $g \in \mathfrak{S}_N$ the Hurwitz action on $\mathrm{Red}_2(g)$ is transitive.*

Sketch of proof

- reduced factorizations of g correspond to maximal chains in $[e, g]_2$
- Proposition 4 implies that it suffices to consider $g = (1\;2\;\ldots\;N)$
- Hurwitz moves on $(1\;2)(2\;3)\cdots(N{-}1\;N) \in \mathrm{Red}_2(g)$ produce reduced factorizations of g starting with $(i\;j)$ for $1 \leq i < j < N$
- apply induction on $\ell_2(g) = N - 1$

Example





Alternating Subgroups of Coxeter Groups

Let (W, S) be a finite Coxeter system with Coxeter matrix $(m_{st})_{s,t \in S}$.

Reflection: any element of $T = \{w^{-1}sw \mid w \in W, s \in S\}$.
Reflection length: $\ell_T(w) = \min\{k \mid w = t_1 t_2 \cdots t_k, t_i \in T\}$.
Alternating subgroup: $\mathfrak{A}(W) = \{w \in W \mid \ell_T(w) \equiv 0 \pmod 2\}$.

The set $A_W = \{w^{-1}stw \mid w \in W, m_{st} \geq 3\}$ generates $\mathfrak{A}(W)$ as a monoid and is closed under W-conjugation. If $W = \mathfrak{S}_N$, then A_W consists of all 3-cycles.

Conjectures in Type B

Let (W, S) be of type B, i.e. W is the hyperoctahedral group of signed permutations. Let $|S| = N$ and $g = (1\;2\;\ldots\;N)(-1\;-2\;\ldots\;-N)$.

Conjecture 11 (Mühle & Nadeau, 2017) *If $N = 2n$, then*

$$\mathcal{Z}_g(m) = \frac{m}{2m - 1}\binom{(2m-1)n}{n}.$$