Ballot-Noncrossing Partitions

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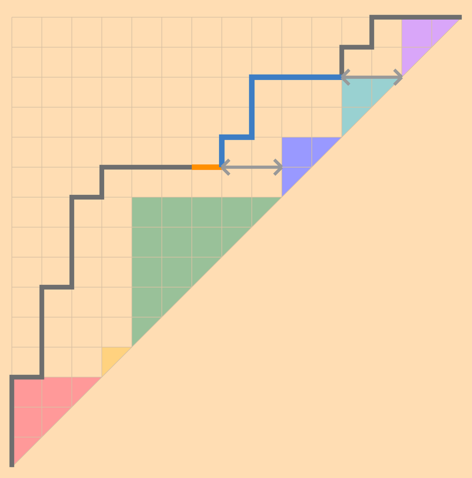
Parabolic Cataland

- generalize Catalan objects subject to a coloring given by an integer composition
- these objects live in parabolic quotients of the symmetric group

Dyck Paths

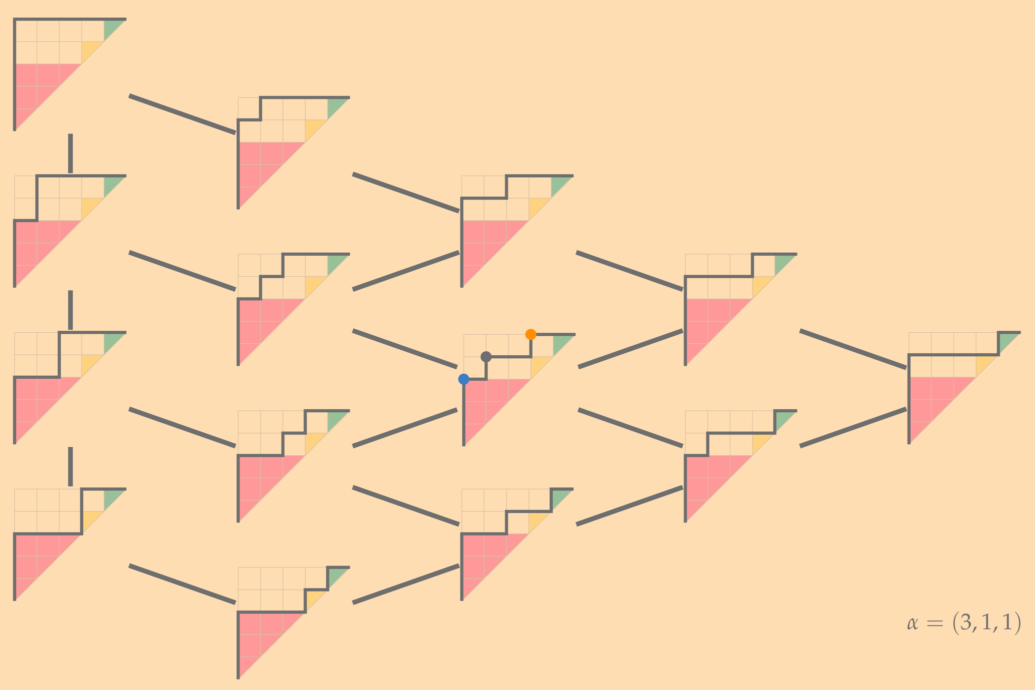
- an **n-Dyck path** is a lattice path from $(0,0)$ to (n,n) that uses only unit north- and east-steps and never passes below the main diagonal
- a **valley** of an n-Dyck path is a subpath EN and a **peak** is a subpath NE
- an **α-Dyck path** is an n-Dyck path that stays weakly above the path $\nu_\alpha \overset{\text{def}}{=} N^{\alpha_1}E^{\alpha_1}N^{\alpha_2}E^{\alpha_2}\dots N^{\alpha_r}E^{\alpha_r}$ $\rightsquigarrow \mathcal{D}_\alpha$



Rotation Order

- a **rotation** of an α-Dyck path μ by a valley EN is the exchange of E with the subpath from N to the next coordinate on μ that has the same horizontal distance to ν_α as the coordinate between E and N
- the **rotation order** on the set of α-Dyck paths is the reflexive and transitive closure of this relation

$\rightsquigarrow \mathrm{Rot}(\alpha)$



$\alpha = (3,1,1)$

Notation

- for a natural number n, let $[n] \overset{\text{def}}{=} \{1,2,\dots,n\}$
- let $\alpha = (\alpha_1,\alpha_2,\dots,\alpha_r)$ be a composition of n
- let $s_i \overset{\text{def}}{=} \alpha_1+\alpha_2+\dots+\alpha_i$ for $i\in[r]$; $s_0 \overset{\text{def}}{=} 0$
- an **α-region** is $\{s_{i-1}+1, s_{i-1}+2,\dots,s_i\}$ for $i\in[r]$

231-Avoiding Permutations

- an **α-permutation** is a permutation w of $[n]$ such that $w(i)<w(i+1)$ for all $i\notin\{s_1,s_2,\dots,s_{r-1}\}$
- a **descent** of an α-permutation w is a pair (i,j) with $i<j$ and $w(i)=w(j)+1$
- an α-permutation w is **$(\alpha,231)$-avoiding** if there do not exist $1\leq i<j<k\leq n$ in different α-regions such that $w(k)<w(i)<w(j)$ and (i,k) is a descent

$\rightsquigarrow \mathfrak{S}_\alpha(231)$

2 11 15 | 1 | 3 7 10 12 13 | 5 6 | 4 9 | 8 14

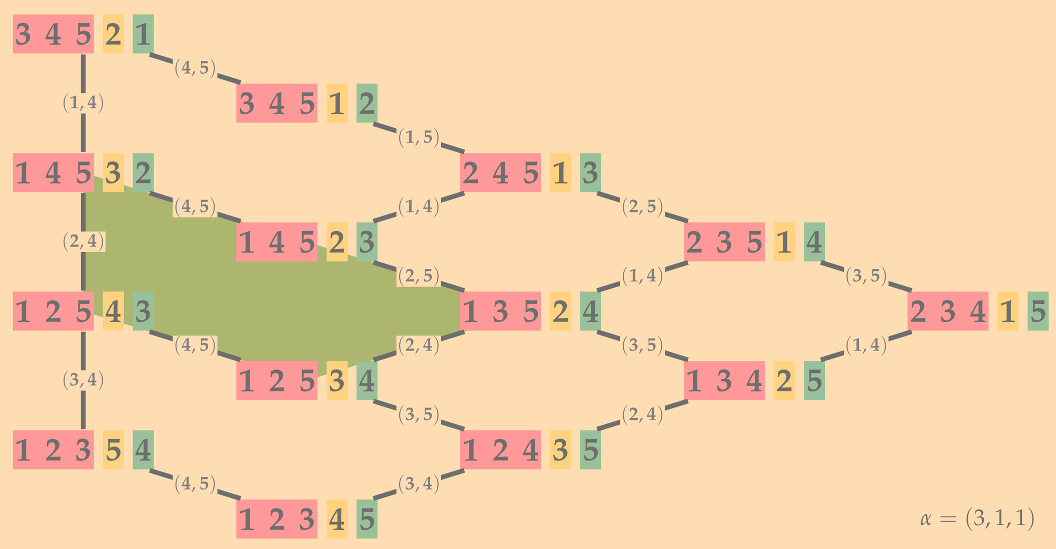
Theorem (& N. Williams, 2015).
For every integer composition α, the sets $\mathcal{D}_\alpha$, $\mathfrak{S}_\alpha(231)$ and NC_α are in bijection.

Theorem (, 2018).
For $n\geq t>0$, the common cardinality of the sets $\mathcal{D}_{\alpha_{(n;t)}}$, $\mathfrak{S}_{\alpha_{(n;t)}}(231)$ and $NC_{\alpha_{(n;t)}}$ is $\frac{t+1}{n+1}\binom{2n-t}{n-t}$.

Weak Order

- an **inversion** of an α-permutation w is a pair (i,j) with $i<j$ and $w(i)>w(j)$
- the **weak order** orders the set of $(\alpha,231)$-avoiding permutations by containment of inversion sets

$\rightsquigarrow \mathrm{Tam}(\alpha)$



$\alpha = (3,1,1)$

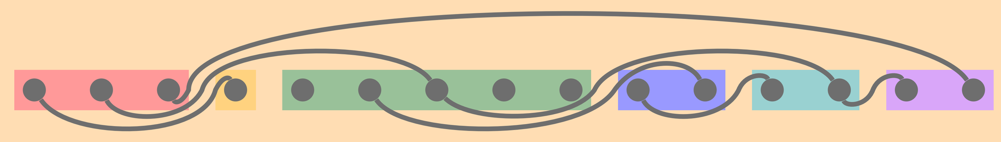
The Ballot Case

- in the case $\alpha = \alpha_{(n;t)} \overset{\text{def}}{=} (t,\underbrace{1,1,\dots,1}_{r\text{ entries}})$, where $n=t+r-1$, we recover ballot paths
- this case generalizes many well-known properties of Catalan objects
- for arbitrary compositions, not all of these generalizations hold

Noncrossing Partitions

- an **α-partition** is a set partition of $[n]$ where no block intersects an α-region more than once
- a **bump** of an α-partition is a pair of consecutive elements in a block
- an α-partition is **noncrossing** if any two distinct bumps (a_1,b_1) and (a_2,b_2) satisfy the following:
  - if $a_1<a_2<b_1<b_2$, then either a_1 and a_2 or b_1 and a_2 belong to the same α-region
  - if $a_1<a_2<b_2<b_1$, then a_1 and a_2 belong to different α-regions

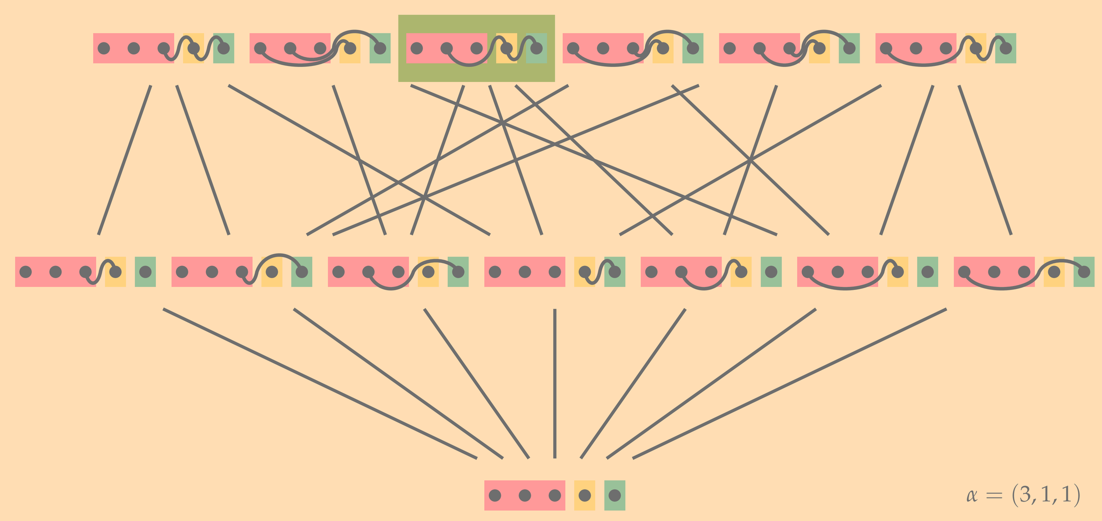
$\rightsquigarrow NC_\alpha$



(Dual) Refinement Order

- an α-partition $\mathbf{P}_1$ **refines** an α-partition $\mathbf{P}_2$ if every block of $\mathbf{P}_1$ is contained in some block of $\mathbf{P}_2$
- the **(dual) refinement order** orders the set of noncrossing α-partitions (dually) by refinement

$\rightsquigarrow \mathrm{Ref}(\alpha)$

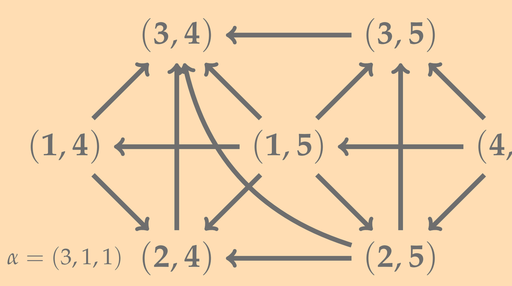


$\alpha = (3,1,1)$

Think: representation of distributive lattices by order ideals of posets.

Galois Graphs

- a finite lattice whose length equals both the number of join- and meet-irreducibles is **extremal**
- extremal lattices can be represented by certain directed graphs; their **Galois graphs**



Think: shard intersection order.

The Core Label Order

- in a finite, edge-labeled lattice, the set of labels between some element x and $x_\downarrow \overset{\text{def}}{=} \bigwedge_{y\lessdot x} y$ is the **core label set** of x
- the **core label order** orders the core label sets by inclusion

Theorem (C. Ceballos, W. Fang, , 2018).
For $n\geq t>0$, the extremal lattices $\mathrm{Rot}(\alpha_{(n;t)})$ and $\mathrm{Tam}(\alpha_{(n;t)})$ admit isomorphic Galois graphs, and are therefore isomorphic.

Recall Wenjie's talk.

Theorem (, 2018).
For $n\geq t>0$, the core label order of $\mathrm{Tam}(\alpha_{(n;t)})$ is isomorphic to $\mathrm{Ref}(\alpha_{(n;t)})$.

Holds for $\alpha=(a,1,\dots,1,b)$.

The H-Triangle

- for $\mu\in\mathcal{D}_{\alpha_{(n;t)}}$ let $\mathrm{p}(\mu)$ denote the number of **peaks**
- let $\mathrm{bo}(\mu)$ be the number of common peaks of μ and $\nu_{\alpha_{(n;t)}}$, and $\mathrm{ba}(\mu)$ be the number of peaks at horizontal distance 1 from $\nu_{\alpha_{(n;t)}}$
- *H-triangle*: $H_{\alpha_{(n;t)}}(p,q) \overset{\text{def}}{=} \sum_{\mu\in\mathcal{D}_{\alpha_{(n;t)}}} p^{\mathrm{p}(\mu)-\mathrm{bo}(\mu)}q^{\mathrm{ba}(\mu)}$

Conjecture (, 2018).
For $n\geq t>0$, we have
$$H_{\alpha_{(n;t)}}(p,q) = \bigl(1+p(q-1)\bigr)^{n-t}M_{\alpha_{(n;t)}}\left(\frac{p(q-1)}{p(q-1)+1},\frac{q}{q-1}\right).$$

Conjectured for $\alpha=(1,\dots,1,a,1,\dots,1)$.

Conjecture (, 2018).
For $n\geq t>0$, the function $F_{\alpha_{(n;t)}}(p,q) = p^{n-t}H_{\alpha_{(n;t)}}\left(\frac{p+1}{p},\frac{q+1}{p+1}\right)$ is a polynomial with nonnegative integer coefficients.

The M-Triangle

- for $\mathbf{P}\in NC_{\alpha_{(n;t)}}$ let $\mathrm{b}(\mathbf{P})$ denote the number of bumps of $\mathbf{P}$
- let μ denote the Möbius function of $\mathrm{Ref}(\alpha_{(n;t)})$
- *M-triangle*: $M_{\alpha_{(n;t)}}(p,q) \overset{\text{def}}{=} \sum_{\mathbf{P}_1,\mathbf{P}_2\in NC_{\alpha_{(n;t)}}} \mu(\mathbf{P}_1,\mathbf{P}_2)p^{\mathrm{b}(\mathbf{P}_2)}q^{\mathrm{b}(\mathbf{P}_1)}$