The Core Label Order of a Congruence-Uniform Lattice

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Outline

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The Doubling Construction

- $\mathcal{P} = (P, \leq)$ finite poset; $X \subseteq P$

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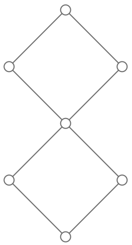

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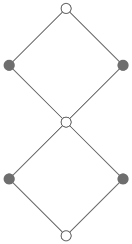

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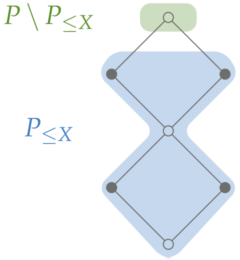

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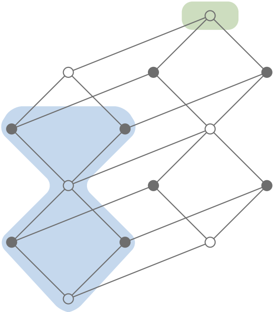

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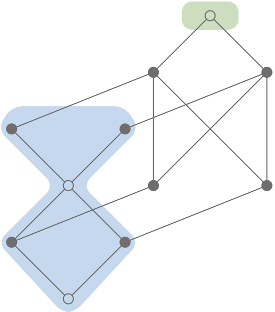

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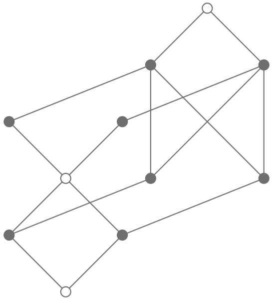

The Doubling Construction

- $\mathcal{P} = (P, \leq)$ finite poset; $X \subseteq P$; $\mathbf{2} = \big(\{0,1\}, \leq\big)$
- **order convex**: for all $x, y, z \in P$ with $x \leq y \leq z$, then $x, z \in X$ implies $y \in X$

Theorem (A. Day; 1970)

If $\mathcal{L} = (L, \leq)$ is a finite lattice and $X \subseteq L$ is order convex, then $\mathcal{L}[X]$ is a lattice, too.

Congruence-Uniform Lattices

- $\mathcal{L} = (L, \leq)$ finite lattice
- **congruence uniform**: can be obtained from the singleton by a sequence of doublings by intervals
- label edges according to creation step $\rightsquigarrow \lambda$

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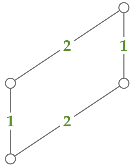

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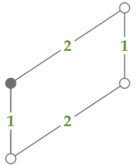

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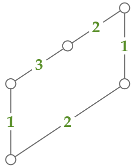

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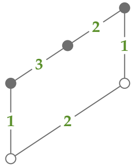

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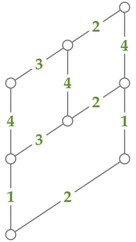

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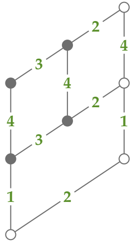

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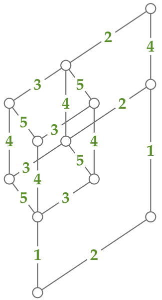

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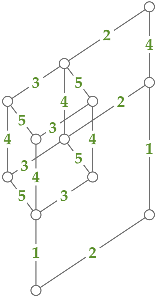

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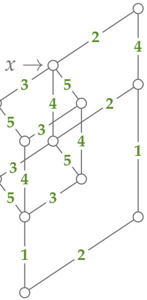

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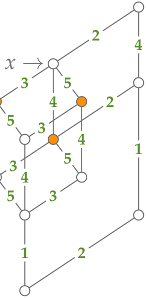

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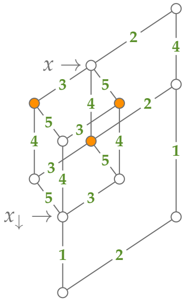

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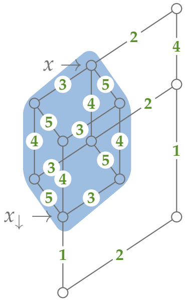

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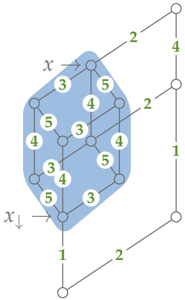

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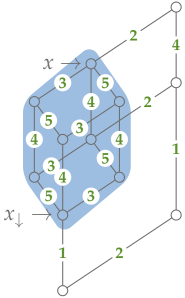

$\Psi(x) = \{3, 4, 5\}$

Congruence-Uniform Lattices

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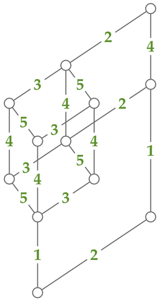

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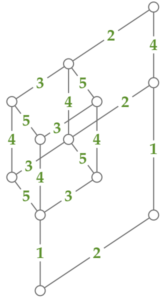

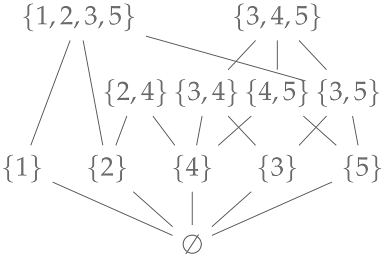

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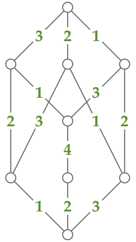

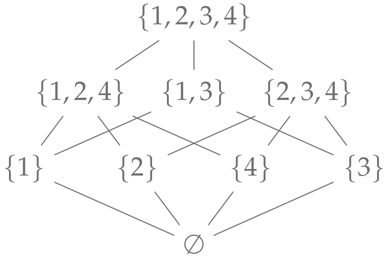

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Question (N. Reading; 2016)

For which congruence-uniform lattices $\mathcal{L}$ is $\mathrm{CLO}(\mathcal{L})$ a lattice, too?

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Spherical Lattices

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- $\mu_{\mathcal{L}}$ Möbius function

Möbius Function

Spherical Lattices

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Theorem (T. McConville; 2015)

If $\mathcal{L}$ is congruence uniform, then $\mu(\mathcal{L}) \in \{-1, 0, 1\}$.

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Theorem (; 2019)

Let $\mathcal{L}$ be congruence uniform. If $\mathrm{CLO}(\mathcal{L})$ is a lattice, then $\mathcal{L}$ is spherical.

The Intersection Property

- $\mathcal{L} = (L, \leq)$ finite lattice
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Theorem (; 2019)

Let $\mathcal{L}$ be congruence uniform. Then, CLO($\mathcal{L}$) *is a meet-semilattice if and only if $\mathcal{L}$ has the intersection property.*

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Corollary (; 2019)

Let $\mathcal{L}$ be congruence uniform. Then, CLO($\mathcal{L}$) *is a lattice if and only if $\mathcal{L}$ is spherical and has the intersection property.*

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Question (; 2019)

Which congruence-uniform lattices have the intersection property?

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A Characterization of Boolean Lattices

- M finite set; $\wp(M)$ power set of M

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Theorem

Every finite Boolean lattice is congruence uniform.

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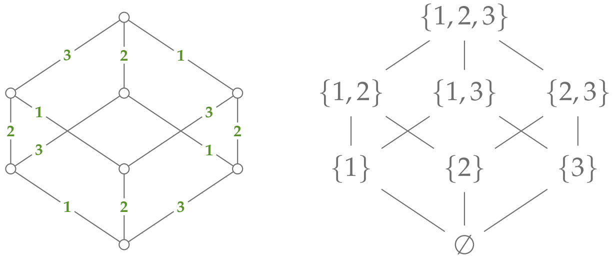

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Theorem (; 2019)

Let $\mathcal{L}$ be congruence uniform. Then, $\mathcal{L} \cong \mathrm{Bool}(M)$ for some finite set M if and only if $\mathcal{L} \cong \mathrm{CLO}(\mathcal{L})$.

Canonical Join Representations

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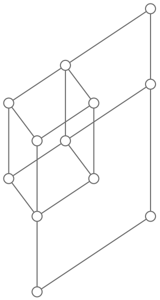

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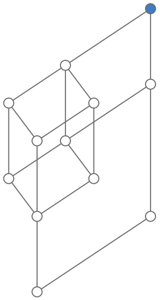

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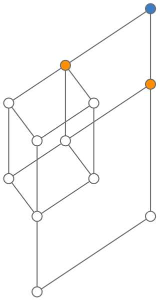

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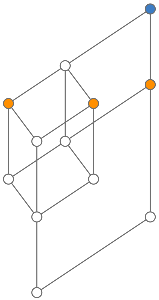

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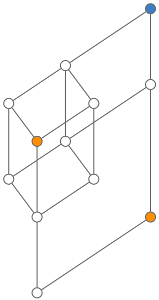

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Theorem (A. Day; 1979)

Every element of a congruence-uniform lattice admits a canonical join representation.

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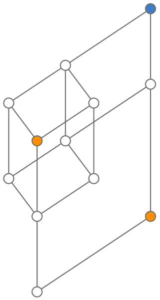

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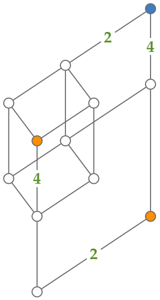

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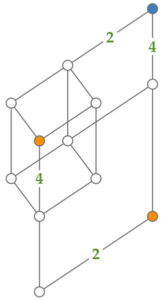
$$\Gamma(x) = \{\lambda(y, x) \mid y \lessdot x\}$$



The Canonical Join Complex

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Proposition (N. Reading; 2015)

For any finite lattice, the set of canonical join representations is closed under taking subsets.

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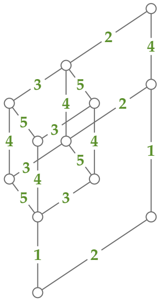

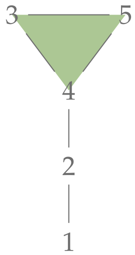

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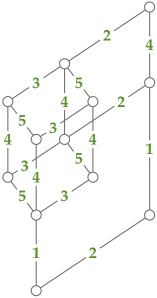

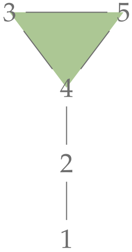

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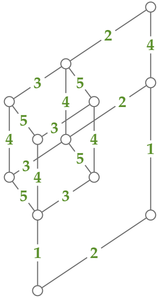

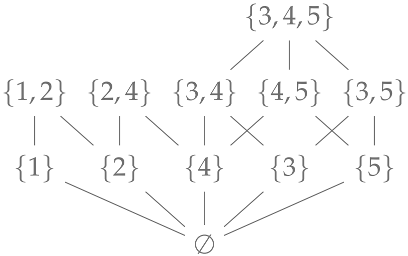

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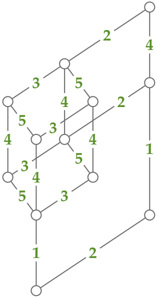

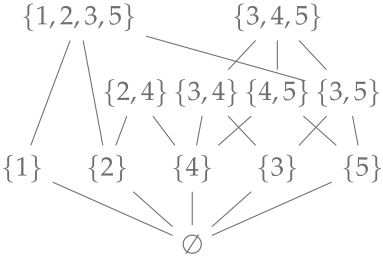

Distributive Lattices

- $\mathcal{L} = (L, \leq)$ finite lattice
- **distributive**: for all $x, y, z \in L$ holds $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and the dual

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Theorem (M. Erné, J. Heitzig, J. Reinhold; 2002)

A finite lattice is distributive if and only if it can be obtained from the singleton lattice by a sequence of doublings by principal order ideals.

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Theorem (; 2018)

A congruence-uniform lattice $\mathcal{L}$ is distributive if and only if $\mathrm{CLO}(\mathcal{L})$ is the face poset of $\mathrm{Can}(\mathcal{L})$.

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Corollary (; 2018)

Every finite distributive lattice has the intersection property.

Thank You.

Möbius Function

- $\mathcal{P} = (P, \leq)$ finite poset
- **Möbius function**: the map $\mu_{\mathcal{P}} : P \times P \to \mathbb{Z}$ given by

$$\mu_{\mathcal{P}}(x,y) = \begin{cases} 1, & \text{if } x = y, \\ -\sum_{x \leq z < y} \mu_{\mathcal{P}}(x,z), & \text{if } x < y, \\ 0, & \text{otherwise} \end{cases}$$

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Theorem (G.-C. Rota; 1964)

Let $\mathcal{P} = (P, \leq)$ be a finite poset, and let $f, g\colon P \times P \to \mathbb{Z}$. It holds $f(y) = \sum_{x \leq y} g(x)$ if and only if $g(y) = \sum_{x \leq y} g(x)\mu_{\mathcal{P}}(x,y)$.

Möbius Function

- $\mathcal{P} = (P, \leq)$ finite bounded poset; $\hat{0}, \hat{1}$ least/greatest element
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Theorem (P. Hall; 1936)

Let $\mathcal{P} = (P, \leq)$ be a finite bounded poset. The reduced Euler characteristic of the order complex of $\big(P \setminus \{\hat{0}, \hat{1}\}, \leq\big)$ equals $\mu_{\mathcal{P}}(\hat{0}, \hat{1})$ up to sign.

The Intersection Property

- $\mathcal{L} = (L, \leq)$ finite lattice

Theorem (; 2019)

Let $\mathcal{L}$ be a finite lattice and let Θ be a lattice congruence of $\mathcal{L}$. If $\mathrm{CLO}(\mathcal{L})$ has the intersection property, then so does $\mathrm{CLO}(\mathcal{L}/\Theta)$.

The Intersection Property

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Theorem (; 2019)

Let $\mathcal{L} = (L, \leq)$ be a finite lattice, and let $x, y \in L$ such that $\Psi(j) \subseteq \Psi(x) \cap \Psi(y)$ for some join-irreducible element $j \in L$. If $j \in [x_{\downarrow}, x] \cap [y_{\downarrow}, y]$. Then $\mathcal{L}[j]$ is spherical if and only if $\mathcal{L}$ is, but $\mathrm{CLO}\big(\mathcal{L}[j]\big)$ is not a meet-semilattice.

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- $\mathcal{L} = (L, \leq)$ finite lattice
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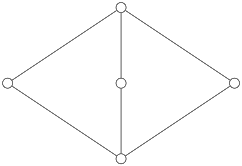

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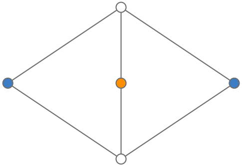

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Theorem

A finite lattice is join-semidistributive if and only if every element admits a canonical join representation.

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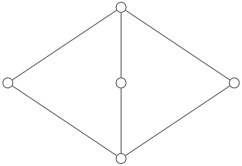

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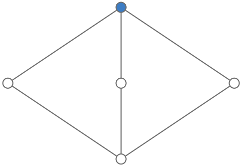

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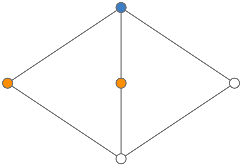

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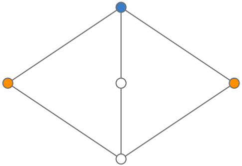

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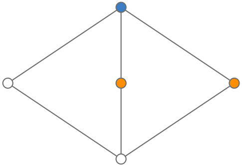

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Theorem (A. Day; 1979)

Every congruence-uniform lattice is semidistributive.