Symmetric Chain Decompositions and the Strong Sperner Property for Noncrossing Partition Lattices

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September 22, 2015

Journées du GT Combinatoire Algébrique du GDR IM

Sperner's Theorem

- $[n] = \{1, 2, \ldots, n\}$ for $n \in \mathbb{N}$
- **antichain**: set of pairwise incomparable subsets of $[n]$

Theorem (E. Sperner, 1928)

The maximal size of an antichain of $[n]$ *is* $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Sperner's Theorem

- ***k*-family**: family of subsets of $[n]$ that can be written as a union of at most k antichains

Theorem (P. Erdős, 1945)

The maximal size of a k-family of $[n]$ is the sum of the k largest binomial coefficients.

A Generalization

- poset perspective:
  - antichain of $[n] \longleftrightarrow$ antichain in the Boolean lattice $\mathcal{B}_n$
  - binomial coefficients $\longleftrightarrow$ rank numbers of $\mathcal{B}_n$
- $\mathcal{P}$ .. graded poset of rank n
- ***k*-Sperner**: size of a k-family does not exceed sum of k largest rank numbers
- **strongly Sperner**: k-Sperner for all $k \leq n$

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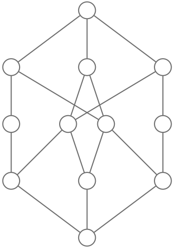
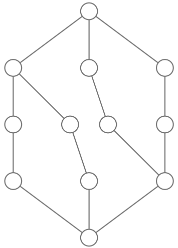
Examples

- a strongly Sperner poset



Examples

- a Sperner poset that is not 2-Sperner



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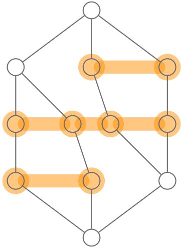

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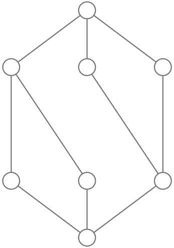

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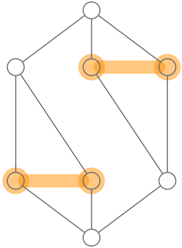

Examples

- strongly Sperner posets:
  - Boolean lattices
  - divisor lattices
  - lattices of noncrossing set partitions
  - Bruhat posets of finite Coxeter groups
  - weak order lattice of H_3
- non-Sperner posets:
  - lattices of set partitions
  - geometric lattices

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- non-Sperner posets:
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  - geometric lattices (certain bond lattices of graphs)

Outline

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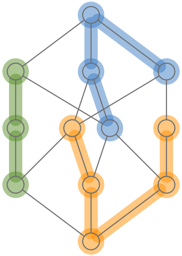
Poset Decompositions

- $\mathcal{P}$ .. graded poset of rank n
- **decomposition**: partition of $\mathcal{P}$ into connected subposets



Poset Decompositions

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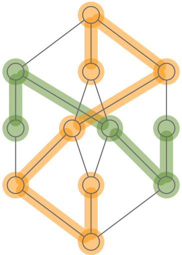

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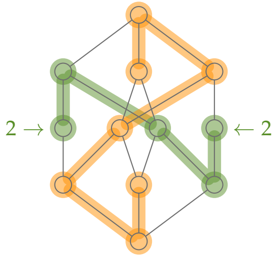

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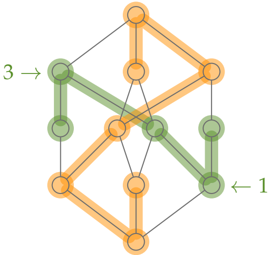

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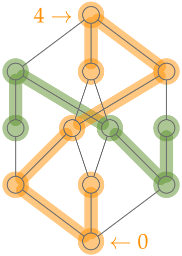

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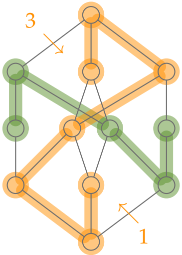
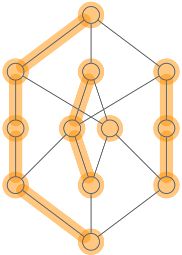


Poset Decompositions

- $\mathcal{P}$ .. graded poset of rank n
- **symmetric chain decomposition**: symmetric decomposition where parts are chains



Symmetric Chain Decompositions

- $\mathcal{P}$ .. graded poset of rank n

Theorem

If $\mathcal{P}$ admits a symmetric chain decomposition, then $\mathcal{P}$ is strongly Sperner.

Symmetric Chain Decompositions

- $\mathcal{P}$ .. graded poset of rank n

Theorem

If $\mathcal{P}$ and $\mathcal{Q}$ admit a symmetric chain decomposition, then so does $\mathcal{P} \times \mathcal{Q}$.

Symmetric Chain Decompositions

- $\mathcal{P}$ .. graded poset of rank n; N_i .. size of i^{th} rank
- **rank-symmetric**: $N_i = N_{n-i}$
- **rank-unimodal**: $N_0 \leq \cdots \leq N_j \geq \cdots \geq N_n$
- **Peck**: strongly Sperner, rank-symmetric, rank-unimodal

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Outline

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Complex Reflection Groups

- V .. n-dimensional unitary vector space
- **(complex) reflection**: unitary transformation of finite order that fixes a hyperplane
- **reflecting hyperplane**: fixed space of a reflection
- **(complex) reflection group**: finite subgroup of $U(V)$ generated by reflections
- **irreducible**: does not preserve a proper subspace of V
- **rank**: codimension of fixed space
- **well-generated**: irreducible, rank equals minimal number of generators
- **parabolic subgroup**: maximal subgroup that fixes a proper subspace of V

Classification of Irreducible Complex Reflection Groups

- one infinite family $G(de,e,n)$:
  - monomial $(n \times n)$-matrices
  - non-zero entries are $(de)^{\text{th}}$ roots of unity
  - product of non-zero entries is d^{th} root of unity
- 34 exceptional groups $G_4, G_5, \ldots, G_{37}$

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- 34 exceptional groups $G_4, G_5, \ldots, G_{37}$
- well-generated complex reflection groups:
  - $G(1,1,n), n \geq 1$
  - $G(d,1,n), d \geq 2, n \geq 1$
  - $G(d,d,n), d,n \geq 2$
  - 26 exceptional groups

Classification of Irreducible Complex Reflection Groups

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- 34 exceptional groups $G_4, G_5, \ldots, G_{37}$
- finite Coxeter groups:
  - $G(1,1,n) \cong A_{n-1}$
  - $G(2,1,n) \cong B_n$
  - $G(2,2,n) \cong D_n$
  - $G(d,d,2) \cong I_2(d)$
  - $G_{24} = H_3, G_{28} = F_4, G_{30} = H_4, G_{35} = E_6, G_{36} = E_7, G_{37} = E_8$

A Distinctive Property

- **degrees**: degrees of a homogeneous choice of generators of the invariant algebra
- usually denoted by $d_1 \leq d_2 \leq \cdots \leq d_n$
- **Coxeter number**: largest degree $h = d_n$

Theorem (G. C. Shephard & J. A. Todd, 1954; C. Chevalley, 1955)

A finite group G is a complex reflection group if and only if its algebra of invariant complex polynomials is a polynomial algebra.

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Regular Elements

- **regular vector**: vector that does not lie in a reflecting hyperplane
- **ζ-regular element**: element with eigenvalue ζ so that the corresponding eigenspace contains a regular vector
- **regular number**: multiplicative order of ζ
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Noncrossing Partitions

- W .. complex reflection group; T .. reflections of W; c .. Coxeter element
- **absolute length**: $\ell_T(w) = \min\{k \mid w = t_1t_2\cdots t_k, t_i \in T\}$
- **absolute order**: $u \leq_T v$ if and only if
$$\ell_T(v) = \ell_T(u) + \ell_T(u^{-1}v)$$
- ***W*-noncrossing partitions**:
$$NC_W(c) = \{w \in W \mid w \leq_T c\}$$
- write $\mathcal{NC}_W(c) = \big(NC_W(c), \leq_T\big)$

Noncrossing Partitions

Theorem (V. Reiner, V. Ripoll & C. Stump, 2015)

For any well-generated complex reflection group W, and any two Coxeter elements $c, c' \in W$ we have $\mathcal{NC}_W(c) \cong \mathcal{NC}_W(c')$.

Theorem (D. Bessis, 2003; D. Bessis, 2015; D. Bessis & R. Corran, 2006; T. Brady, 2001; T. Brady & C. Watt, 2002; T. Brady & C. Watt, 2008; G. Kreweras, 1972; V. Reiner, 1997)

The poset $\mathcal{NC}_W$ is a lattice for any well-generated complex reflection group W.

Catalan Numbers

- ***W*-Catalan number**:

$$\mathrm{Cat}_W = \prod_{i=1}^{n} \frac{d_i + h}{d_i}$$

Theorem (C. A. Athanasiadis & V. Reiner, 2004; D. Bessis, 2003; D. Bessis, 2015; D. Bessis & R. Corran, 2006; G. Kreweras, 1972; V. Reiner, 1997)

We have $\left|NC_W\right| = Cat_W$ for any well-generated complex reflection group W.

Outline

Symmetric Chain Decompositions of $\mathcal{NC}_{G(1,1,n)}$

- $W = G(1,1,n) \cong \mathfrak{S}_n$; T .. transpositions; $c = (1\ 2\ \ldots\ n)$
- $\mathcal{NC}_{G(1,1,n)}(c)$ is isomorphic to the lattice of noncrossing set partitions of $[n]$
- $R_k = \{w \in NC_{G(1,1,n)}(c) \mid w(1) = k\}$, $\mathcal{R}_k = (R_k, \leq_T)$

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- $\uplus$ .. disjoint set union; $\mathbf{2}$ .. 2-chain

Lemma (R. Simion & D. Ullmann, 1991)

We have $\mathcal{R}_1 \uplus \mathcal{R}_2 \cong \mathbf{2} \times \mathcal{NC}_{G(1,1,n-1)}$, and $\mathcal{R}_i \cong \mathcal{NC}_{G(1,1,i-2)} \times \mathcal{NC}_{G(1,1,n-i+1)}$ whenever $3 \leq i \leq n$. Moreover, this decomposition is symmetric.

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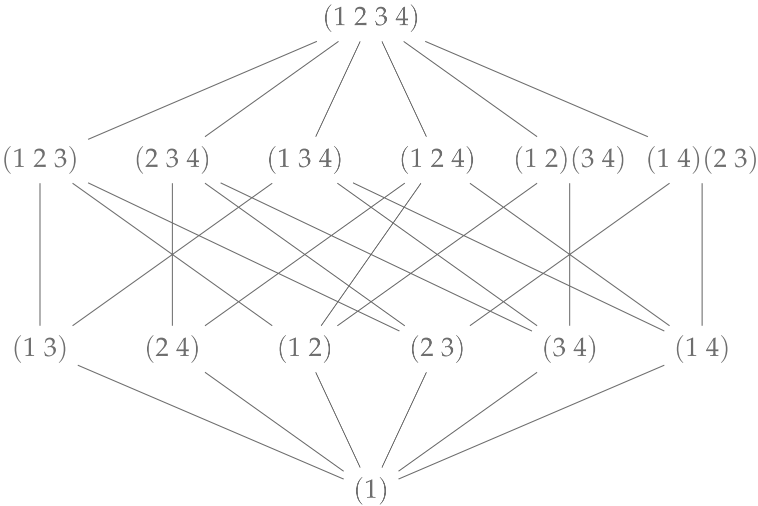
Theorem (R. Simion & D. Ullmann, 1991)

The lattice $\mathcal{NC}_{G(1,1,n)}$ admits a symmetric chain decomposition for each $n \geq 1$.

Example: $\mathcal{NC}_{\mathfrak{S}_4}\big((1\,2\,3\,4)\big)$



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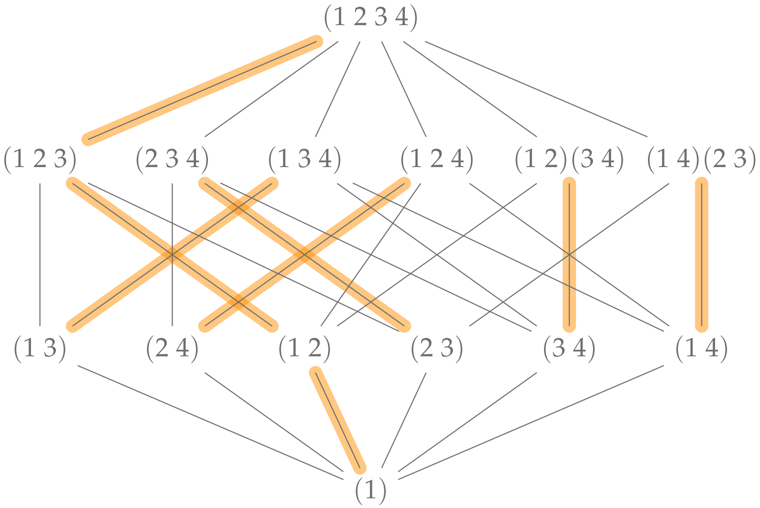

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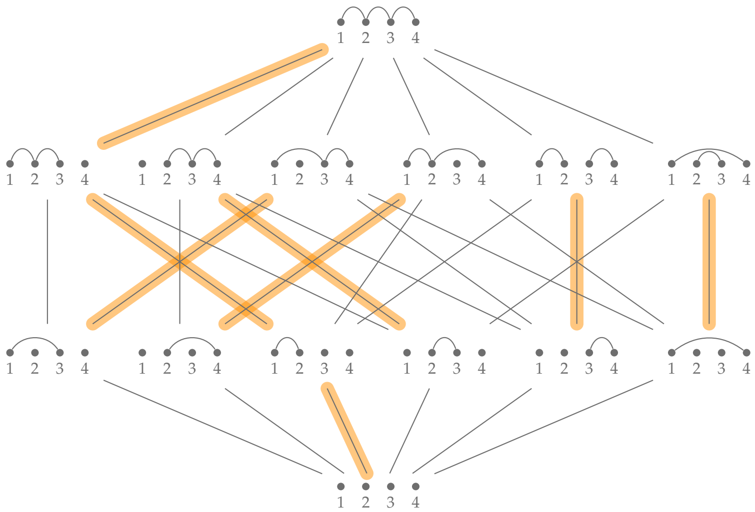

Outline

The Groups $G(d,d,n)$, $d,n \geq 2$

- subgroups of $\mathfrak{S}_{dn}$, permuting elements of

$$\left\{1^{(0)},\ldots,n^{(0)},1^{(1)},\ldots,n^{(1)},\ldots,1^{(d-1)},\ldots,n^{(d-1)}\right\}$$

- $w \in G(d,d,n)$ satisfies $w\left(k^{(s)}\right) = \pi(k)^{(s+t_k)}$
  - $\sum_{k=1}^{n} t_k \equiv 0 \pmod d$
  - $\pi \in \mathfrak{S}_n$, and t_k depends on w and k

The Groups $G(d,d,n)$, $d,n \geq 2$

- elements can be decomposed into "cycles":

$$\Big(\Big(k_1^{(t_1)} \; \ldots \; k_r^{(t_r)}\Big)\Big) = \Big(k_1^{(t_1)} \; \ldots \; k_r^{(t_r)}\Big)\Big(k_1^{(t_1+1)} \; \ldots \; k_r^{(t_r+1)}\Big) \cdots \Big(k_1^{(t_1+d-1)} \; \ldots \; k_r^{(t_r+d-1)}\Big),$$

and

$$\Big[k_1^{(t_1)} \; \ldots \; k_r^{(t_r)}\Big]_s = \Big(k_1^{(t_1)} \; \ldots \; k_r^{(t_r)} \; k_1^{(t_1+s)} \; \ldots \; k_r^{(t_r+s)} \; \ldots \; k_1^{(t_1(d-1)s)} \; \ldots \; k_r^{(t_r+(d-1)s)}\Big).$$

The Lattices $\mathcal{NC}_{G(d,d,n)}$, $d,n \geq 2$

- Coxeter element $c = \left[1^{(0)} \;\ldots\; (n-1)^{(0)}\right]_1 \left[n^{(0)}\right]_{d-1}$
- matrix representation:

$$c = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \zeta_d & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \zeta_d^{d-1} \end{pmatrix},$$

where $\zeta = e^{2\pi\sqrt{-1}/d}$

The Lattices $\mathcal{NC}_{G(d,d,n)}$, $d,n \geq 2$

Proposition (✻, 2015)

For $d,n \geq 2$, the atoms in $\mathcal{NC}_{G(d,d,n)}(c)$ are of one of the following forms:

- $\left(\left(a^{(0)}\ b^{(s)}\right)\right)$ *for* $1 \leq a < b < n$ *and* $s \in \{0, d-1\}$, *or*
- $\left(\left(a^{(0)}\ n^{(s)}\right)\right)$ *for* $1 \leq a < n$ *and* $0 \leq s < d$.

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Proposition (, 2015)

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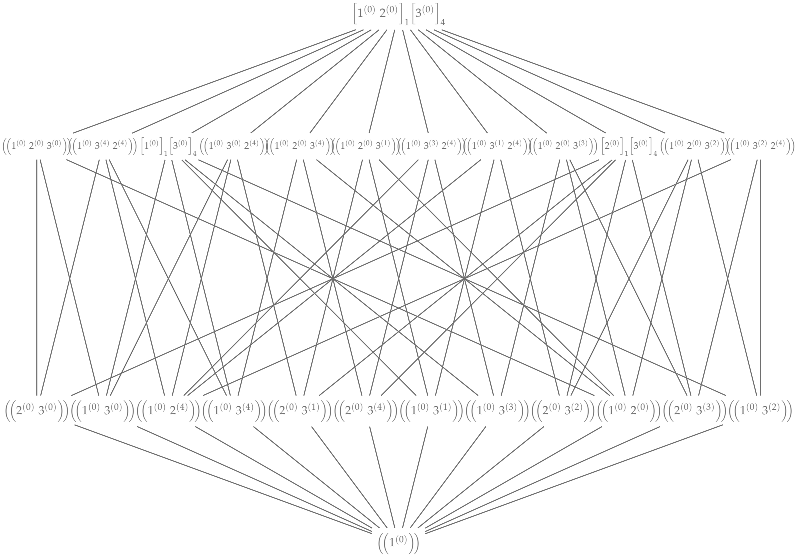
- $\left[1^{(0)}\ \ldots\ a^{(0)}\ (b+1)^{(0)}\ \ldots\ (n-1)^{(0)}\right]_1 \left[n^{(0)}\right]_{d-1}$ $\left(\left((a+1)^{(0)}\ \ldots\ b^{(0)}\right)\right)$ *for* $1 \leq a < b < n$,
- $\left(\left(1^{(0)}\ \ldots\ a^{(0)}\ (b+1)^{(d-1)}\ \ldots\ (n-1)^{(d-1)}\right)\right)$ $\left[(a+1)^{(0)}\ \ldots\ b^{(0)}\right]_1 \left[n^{(0)}\right]_{d-1}$ *for* $1 \leq a < b < n$, *or*
- $\left(\left(1^{(0)}\ \ldots\ a^{(0)}\ n^{(s-1)}\ (a+1)^{(d-1)}\ \ldots\ (n-1)^{(d-1)}\right)\right)$ *for* $1 \leq a < n$ *and* $0 \leq s < d$.

Example: $d = 5, n = 3$



Outline

A First Decomposition

- $R_k^{(s)} = \left\{w \in NC_{G(d,d,n)}(c) \mid w\left(1^{(0)}\right) = k^{(s)}\right\}$
- $\mathcal{R}_k^{(s)} = \left(R_k^{(s)}, \leq_T\right)$

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Lemma (, 2015)

The sets $R_1^{(s)}$ *and* $R_k^{(s')}$ *are empty for* $2 \leq s < d$ *as well as* $2 \leq k < n$ *and* $1 \leq s' < d-1$.

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Lemma (🌬, 2015)

The poset $\mathcal{R}_1^{(0)} \uplus \mathcal{R}_2^{(0)}$ *is isomorphic to* $\mathbf{2} \times \mathcal{NC}_{G(d,d,n-1)}$. *Moreover, its least element has length* 0*, and its greatest element has length* n*.*

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Lemma (, 2015)

The poset $\mathcal{R}_n^{(s)}$ is isomorphic to $\mathcal{NC}_{G(1,1,n-1)}$ for $0 \leq s < d$. Moreover, its least element has length 1, and its greatest element has length $n-1$.

A First Decomposition

- $R_k^{(s)} = \left\{ w \in NC_{G(d,d,n)}(c) \mid w\left(1^{(0)}\right) = k^{(s)} \right\}$
- $\mathcal{R}_k^{(s)} = \left(R_k^{(s)}, \leq_T \right)$

Lemma (, 2015)

The poset $\mathcal{R}_i^{(0)}$ is isomorphic to $\mathcal{NC}_{G(d,d,n-i+1)} \times \mathcal{NC}_{G(1,1,i-2)}$ whenever $3 \leq i < n$. Moreover, its least element has length 1, and its greatest element has length $n-1$.

A First Decomposition

- $R_k^{(s)} = \Big\{ w \in NC_{G(d,d,n)}(c) \mid w\big(1^{(0)}\big) = k^{(s)} \Big\}$
- $\mathcal{R}_k^{(s)} = \big(R_k^{(s)}, \leq_T \big)$

Lemma (, 2015)

The poset $\mathcal{R}_i^{(d-1)}$ is isomorphic to $\mathcal{NC}_{G(1,1,n-i)} \times \mathcal{NC}_{G(d,d,i-1)}$ whenever $3 \leq i < n$. Moreover, its least element has length 1, and its greatest element has length $n-1$.

A First Decomposition

- $R_k^{(s)} = \left\{ w \in NC_{G(d,d,n)}(c) \mid w\left(1^{(0)}\right) = k^{(s)} \right\}$
- $\mathcal{R}_k^{(s)} = \left(R_k^{(s)}, \leq_T \right)$

Lemma (🏠, 2015)

The poset $\mathcal{R}_1^{(1)}$ is isomorphic to $\mathcal{NC}_{G(1,1,n-2)}$. Moreover, its least element has length 2, and its greatest element has length $n-1$.

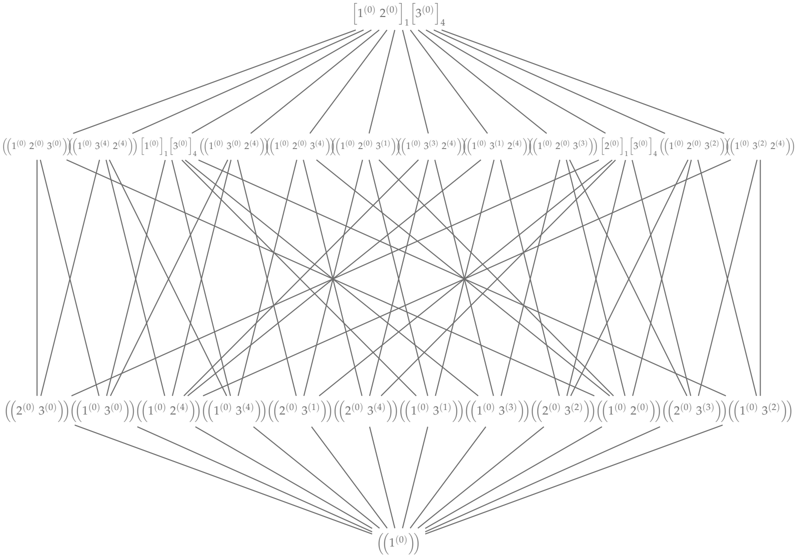
A First Decomposition

- $R_k^{(s)} = \Big\{ w \in NC_{G(d,d,n)}(c) \mid w\Big(1^{(0)}\Big) = k^{(s)} \Big\}$
- $\mathcal{R}_k^{(s)} = \Big(R_k^{(s)}, \leq_T \Big)$

Lemma (, 2015)

The poset $\mathcal{R}_2^{(d-1)}$ *is isomorphic to* $\mathcal{NC}_{G(1,1,n-2)}$. *Moreover, its least element has length* 1, *and its greatest element has length* $n-2$.

Example: $d = 5, n = 3$



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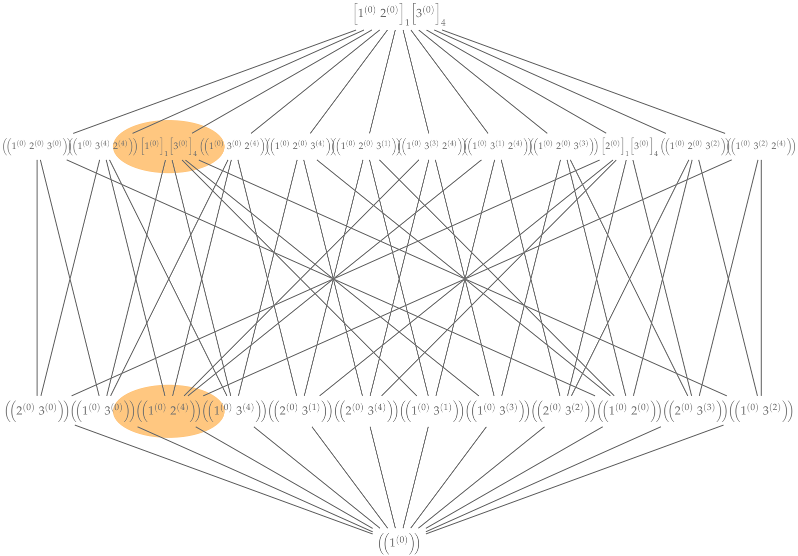

Outline

A Second Decomposition

- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$

A Second Decomposition

- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$
- consider the map

  $f_1 : R_1^{(1)} \to NC_{G(d,d,n)}(c), \quad x \mapsto \Big(\big(1^{(0)}\ n^{(d-2)}\big)\Big)x$

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  $f_1 : R_1^{(1)} \to R_n^{(d-1)}, \quad x \mapsto \Big(\big(1^{(0)}\ n^{(d-2)}\big)\Big)x$
- this map is an injective involution
- its image consists of permutations $w \in R_n^{(d-1)}$ with $w\Big(n^{(d-1)}\Big) = 1^{(0)}$

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- this map is an injective involution
- its image is the interval
  $\left[\left(\left(1^{(0)}\ n^{(d-1)}\right)\right), \left(\left(1^{(0)}\ n^{(d-1)}\right)\right)\left(\left(2^{(0)}\ \ldots\ (n-1)^{(0)}\right)\right)\right]_T$

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Lemma (, 2015)

The interval $\left(f_1\left(R_1^{(1)}\right), \leq_T\right)$ *is isomorphic to* $\mathcal{NC}_{G(1,1,n-2)}$.

A Second Decomposition

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- consider the map
  $f_1 : R_1^{(1)} \to R_n^{(d-1)}, \quad x \mapsto \Big(\big(1^{(0)}\ n^{(d-2)}\big)\Big)x$
- define $D_1 = R_1^{(1)} \uplus f_1\left(R_1^{(1)}\right)$, and $\mathcal{D}_1 = (D_1, \leq_T)$

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Lemma (, 2015)

The poset $\mathcal{D}_1$ is isomorphic to $\mathbf{2} \times \mathcal{NC}_{G(1,1,n-2)}$. Moreover, its least element has length 1, and its greatest element has length $n-1$.

A Second Decomposition

- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$
- consider the map

$$f_2 : R_2^{(d-1)} \to NC_{G(d,d,n)}(c), \quad x \mapsto \Big(\big(2^{(0)}\ n^{(0)}\big)\Big)x$$

A Second Decomposition

- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$
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- its image consists of permutations $w \in R_n^{(d-1)}$ with $w\left(n^{(d-1)}\right) = 2^{(d-1)}$

A Second Decomposition

- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$
- consider the map
  $$f_2 : R_2^{(d-1)} \to R_n^{(d-1)}, \quad x \mapsto \Big(\big(2^{(0)}\ n^{(0)}\big)\Big)x$$
- this map is an injective involution
- its image is the interval

$$\Big[\Big(\big(1^{(0)}\ n^{(d-1)}\ 2^{(d-1)}\big)\Big), \Big(\big(1^{(0)}\ n^{(d-1)}\ 2^{(d-1)}\ \ldots\ (n-1)^{(d-1)}\big)\Big)\Big]_T$$

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The interval $\Big(f_2\big(R_2^{(d-1)}\big), \leq_T\Big)$ *is isomorphic to* $\mathcal{NC}_{G(1,1,n-2)}$.

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A Second Decomposition

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Lemma (, 2015)

The poset $\mathcal{D}_2$ is isomorphic to $\mathbf{2} \times \mathcal{NC}_{G(1,1,n-2)}$. Moreover, its least element has length 1, and its greatest element has length $n-1$.

A Second Decomposition

- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$
- define $D = R_n^{(d-1)} \setminus \left(f_1\left(R_1^{(1)}\right) \uplus f_2\left(R_2^{(d-1)}\right)\right)$, and $\mathcal{D} = (D, \leq_T)$

Lemma (, 2015)

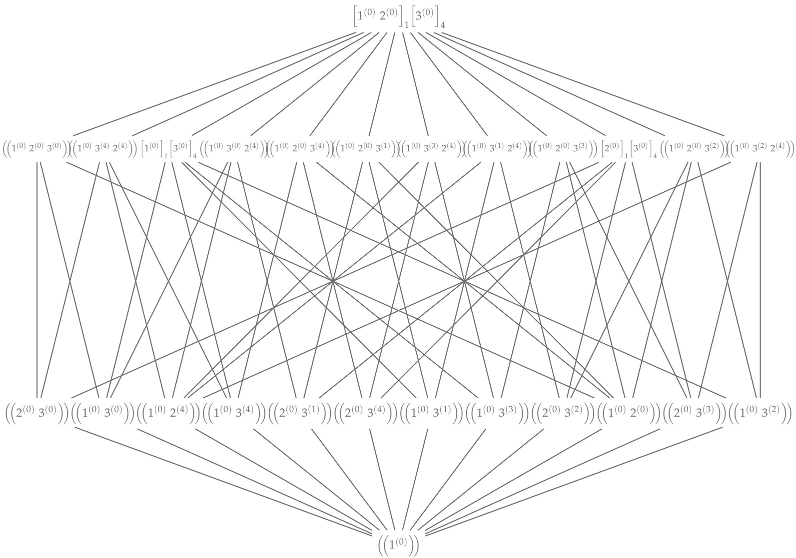
The poset $\mathcal{D}$ is isomorphic to $\biguplus_{i=3}^{n-1} \mathcal{NC}_{G(1,1,i-2)} \times \mathcal{NC}_{G(1,1,n-i)}$. Morever, its minimal elements have length 2, and its maximal elements have length $n-2$.

The Main Result

Theorem (, 2015)

For $d, n \geq 2$ the lattice $\mathcal{NC}_{G(d,d,n)}$ admits a symmetric chain decomposition. Consequently, it is Peck.

Example: $d = 5, n = 3$



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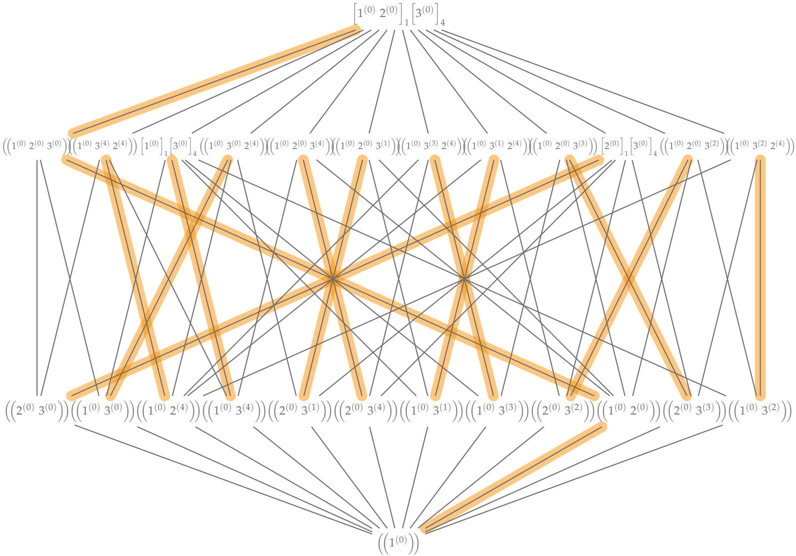

Outline

The Remaining Cases

- so far: $\mathcal{NC}_{G(1,1,n)}$ and $\mathcal{NC}_{G(d,d,n)}$ admit symmetric chain decompositions
- what about the other well-generated complex reflection groups?

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The lattice $\mathcal{NC}_{G(2,1,n)}$ admits a symmetric chain decomposition for any $n \geq 1$.

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- what about the other well-generated complex reflection groups?
- we have $\mathcal{NC}_{G(2,1,n)} \cong \mathcal{NC}_{G(d,1,n)}$ for $d \geq 2$ and $n \geq 1$
- only the 26 exceptional groups remain

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A Decomposition Argument

- $\mathcal{P}$ .. graded poset of rank n
- $\mathcal{P}[i]$ .. subposet of $\mathcal{P}$ with i largest ranks removed

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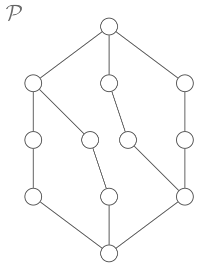

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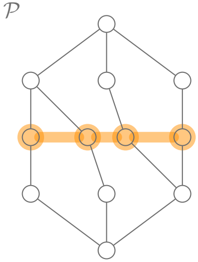

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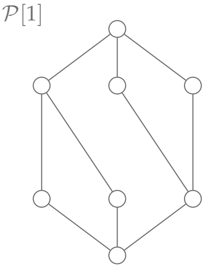

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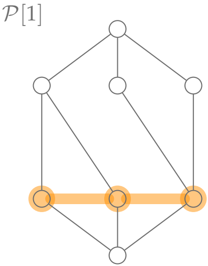

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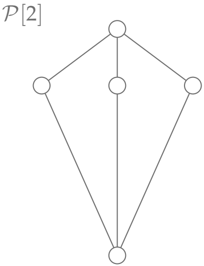

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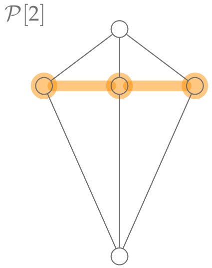

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$\mathcal{P}[3]$

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$\mathcal{P}[4]$

A Decomposition Argument

- $\mathcal{P}$ .. graded poset of rank n
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Proposition (, 2015)

A graded poset $\mathcal{P}$ of rank n is strongly Sperner if and only if $\mathcal{P}[i]$ is Sperner for all $i \in \{0, 1, \ldots, n\}$.

- antichains in $\mathcal{P}[i]$ are antichains in $\mathcal{P}[s]$ for $s < i$

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A Decomposition Argument

- SAGE has a fast implementation to compute the size of the largest antichain of a poset

A Decomposition Argument

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The lattice $\mathcal{NC}_W$ is Peck for any well-generated exceptional complex reflection group W.

A Decomposition Argument

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m-Divisible Noncrossing Partition Posets

- W .. well-generated complex reflection group; c .. Coxeter element of W
- **m-divisible noncrossing partition**: m-multichain of noncrossing partitions $\rightsquigarrow NC_W^{(m)}(c)$

$$(w)_m = (w_1, w_2, \ldots, w_m) \text{ with } w_1 \leq_T w_2 \leq_T \cdots \leq_T w_m \leq_T c$$

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- ***m*-delta sequence**: sequence of "differences" of elements in a multichain

$$(w)_m = (w_1, w_2, \ldots, w_m) \text{ with } w_1 \leq_T w_2 \leq_T \cdots \leq_T w_m \leq_T c$$

$$\partial(w)_m = \left[w_1; w_1^{-1}w_2, w_2^{-1}w_3, \ldots, w_{m-1}^{-1}w_m, w_m^{-1}c\right]$$

m-Divisible Noncrossing Partition Posets

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- ***m*-divisible noncrossing partition**: m-multichain of noncrossing partitions $\rightsquigarrow NC_W^{(m)}(c)$
- ***m*-delta sequence**: sequence of "differences" of elements in a multichain
- partial order: $(u)_m \leq (v)_m$ if and only if $\partial(u)_m \leq_T \partial(v)_m$ $\rightsquigarrow \mathcal{NC}_W^{(m)}(c)$

Question (D. Armstrong, 2009)

Are the posets $\mathcal{NC}_W^{(m)}$ strongly Sperner for any W and any $m \geq 1$?

m-Divisible Noncrossing Partition Posets

- affirmative answer for $m = 1$

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m-Divisible Noncrossing Partition Posets

- affirmative answer for $m = 1$
- what about $m > 1$?
  - $\mathcal{NC}_W^{(m)}$ is antiisomorphic to an order ideal in $(\mathcal{NC}_W)^m$
  - $(\mathcal{NC}_W)^m$ is Peck
  - $\mathcal{NC}_W^{(m)}$ is not rank-symmetric $\rightsquigarrow$ no symmetric chain decomposition

Question (D. Armstrong, 2009)

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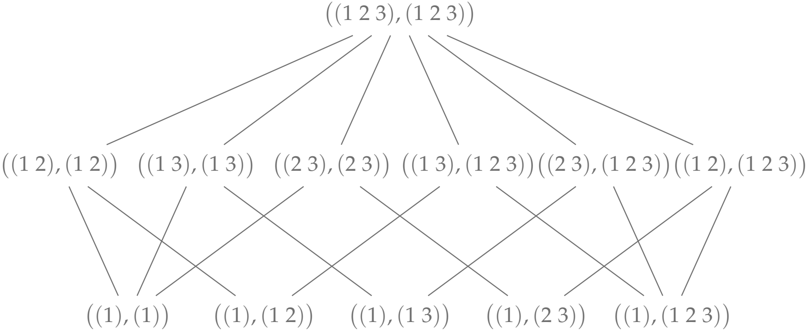
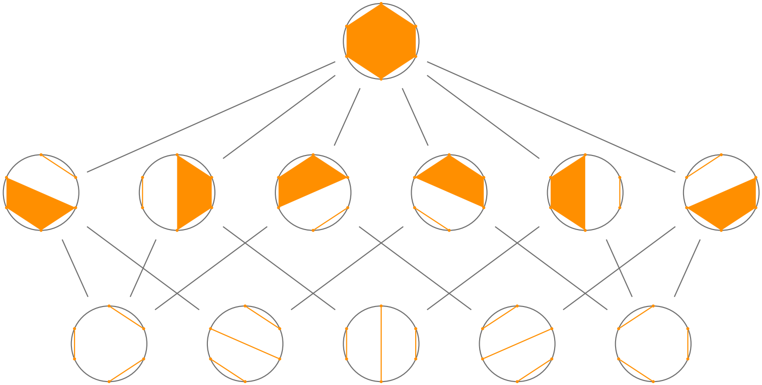
Example: $\mathcal{NC}^{(2)}_{\mathfrak{S}_4}$



Example: $\mathcal{NC}^{(2)}_{\mathfrak{S}_4}$



Thank You.

Interlude: A Convolution Formula

$$NC_{G(d,d,n)}(c) = R_1^{(0)} \uplus R_1^{(1)} \uplus \biguplus_{i=2}^{n-1} \left(R_i^{(0)} \uplus R_i^{(d-1)}\right) \uplus \biguplus_{s=0}^{d-1} R_n^{(s)}$$

Interlude: A Convolution Formula

Proposition (🏠, 2015)

For $n \geq 0$ we have $\sum_{i=0}^{n} i \cdot Cat_{G(1,1,i)} \cdot Cat_{G(1,1,n-i)} = \binom{2n+1}{n-1}$.

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$$\mathrm{Cat}_{G(d,d,n+2)} = 2 \cdot \mathrm{Cat}_{G(d,d,n+1)} + 2 \cdot \mathrm{Cat}_{G(1,1,n)} + d \cdot \mathrm{Cat}_{G(1,1,n+1)}$$
$$+ 2 \sum_{i=3}^{n+1} \mathrm{Cat}_{G(d,d,n-i+3)})\mathrm{Cat}_{G(1,1,i-2)}$$

Interlude: A Convolution Formula

Proposition (🏛, 2015)

For $n \geq 0$ we have $\sum_{i=0}^{n} i \cdot Cat_{G(1,1,i)} \cdot Cat_{G(1,1,n-i)} = \binom{2n+1}{n-1}$.

$$\mathrm{Cat}_{G(d,d,n+2)} = d \cdot \mathrm{Cat}_{G(1,1,n+1)} + 2 \cdot \sum_{i=2}^{n+2} \mathrm{Cat}_{G(d,d,n-i+3)} \cdot \mathrm{Cat}_{G(1,1,i-2)}$$

Interlude: A Convolution Formula

Proposition (🌬, 2015)

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$$\mathrm{Cat}_{G(d,d,n+2)} = \left(\prod_{i=1}^{n+1} \frac{di + (n-1)d}{di}\right) \frac{n + (n-1)d}{n}$$

Interlude: A Convolution Formula

Proposition (, 2015)

For $n \geq 0$ we have $\sum_{i=0}^{n} i \cdot Cat_{G(1,1,i)} \cdot Cat_{G(1,1,n-i)} = \binom{2n+1}{n-1}$.

$$\begin{aligned} \mathrm{Cat}_{G(d,d,n+2)} &= d \cdot \mathrm{Cat}_{G(1,1,n+1)} \\ &\qquad + 2 \cdot \sum_{i=0}^{n} \mathrm{Cat}_{G(d,d,i+1)} \cdot \mathrm{Cat}_{G(1,1,n-i)} \end{aligned}$$

$$\mathrm{Cat}_{G(d,d,n+2)} = \Big((n+1)d + n + 2\Big) \cdot \mathrm{Cat}_{G(1,1,n+1)}$$

Interlude: A Convolution Formula

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For $n \geq 0$ we have $\displaystyle\sum_{i=0}^{n} i \cdot Cat_{G(1,1,i)} \cdot Cat_{G(1,1,n-i)} = \binom{2n+1}{n-1}$.

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Interlude: A Convolution Formula

Proposition (, 2015)

For $n \geq 0$ we have $\sum_{i=0}^{n} i \cdot Cat_{G(1,1,i)} \cdot Cat_{G(1,1,n-i)} = \binom{2n+1}{n-1}$.

$$nd \cdot \mathrm{Cat}_{G(1,1,n+1)} + \binom{2(n+1)}{n+1} = 2 \cdot \sum_{i=0}^{n} (id + i + 1) \cdot \mathrm{Cat}_{G(1,1,i)} \cdot \mathrm{Cat}_{G(1,1,n-i)}$$

Interlude: A Convolution Formula

Proposition (, 2015)

For $n \geq 0$ we have $\sum_{i=0}^{n} i \cdot Cat_{G(1,1,i)} \cdot Cat_{G(1,1,n-i)} = \binom{2n+1}{n-1}$.

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$$2 \cdot \sum_{i=0}^{n} \Big(id \cdot \mathrm{Cat}_{G(1,1,i)} \cdot \mathrm{Cat}_{G(1,1,n-i)} \Big) + 2 \cdot \sum_{i=0}^{n} \binom{2i}{i} \cdot \mathrm{Cat}_{G(1,1,n-i)}$$

Interlude: A Convolution Formula

Proposition (windmill, 2015)

For $n \geq 0$ we have $$\sum_{i=0}^{n} i \cdot Cat_{G(1,1,i)} \cdot Cat_{G(1,1,n-i)} = \binom{2n+1}{n-1}.$$

$$nd \cdot \mathrm{Cat}_{G(1,1,n+1)} + \binom{2(n+1)}{n+1} = 2d \cdot \sum_{i=0}^{n} \left(i \cdot \mathrm{Cat}_{G(1,1,i)} \cdot \mathrm{Cat}_{G(1,1,n-i)} \right) + \binom{2(n+1)}{n+1}$$

Interlude: A Convolution Formula

Proposition (🌬, 2015)

For $n \geq 0$ *we have* $\sum_{i=0}^{n} i \cdot Cat_{G(1,1,i)} \cdot Cat_{G(1,1,n-i)} = \binom{2n+1}{n-1}$.

$$\frac{n}{2} \cdot \mathrm{Cat}_{G(1,1,n+1)} = \sum_{i=0}^{n} i \cdot \mathrm{Cat}_{G(1,1,i)} \cdot \mathrm{Cat}_{G(1,1,n-i)}$$

Interlude: A Convolution Formula

Proposition (2015)

For $n \geq 0$ we have $\displaystyle\sum_{i=0}^{n} i \cdot Cat_{G(1,1,i)} \cdot Cat_{G(1,1,n-i)} = \binom{2n+1}{n-1}$.

$$\binom{2n+1}{n-1} = \sum_{i=0}^{n} i \cdot \mathrm{Cat}_{G(1,1,i)} \cdot \mathrm{Cat}_{G(1,1,n-i)}$$

Interlude: A Convolution Formula

Proposition (Y. Kong, 2000)

For $n \geq 0$ we have $\displaystyle\sum_{i=0}^{n-1} Cat_{G(1,1,i)} \cdot \binom{2(n-i)}{n-i-1} = \binom{2n+1}{n-1}$.