

Posets from Polytopes and Refined Face Enumeration

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Outline

Posets from
Polytopes and
Refined Face
Enumeration

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Posets from
Polytopes

Refined Face
Enumeration

Chapoton's F -
and
 H -Triangle

ν -
Associahedra

The $F=H$ -
Correspondence

- 1 Posets from Polytopes
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- 4 ν -Associahedra
- 5 The $F=H$ -Correspondence

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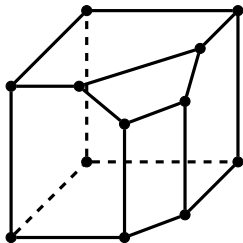
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- $\mathcal{P} \subseteq \mathbb{R}^d$.. polytope



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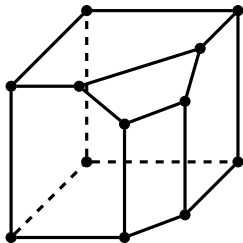
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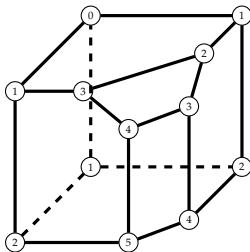
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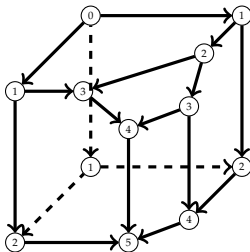
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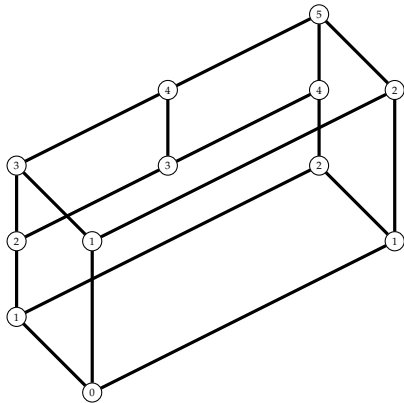
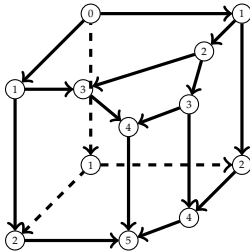
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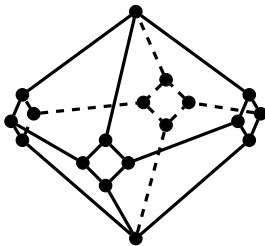
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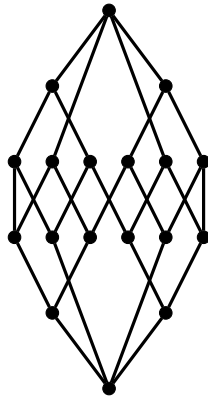
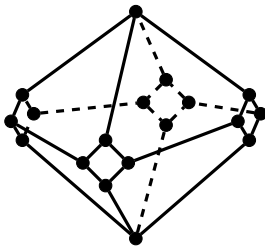
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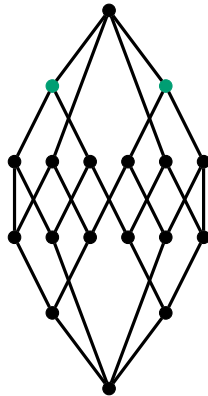
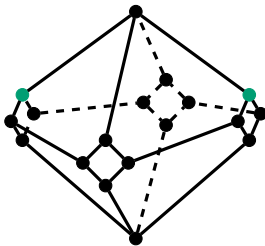
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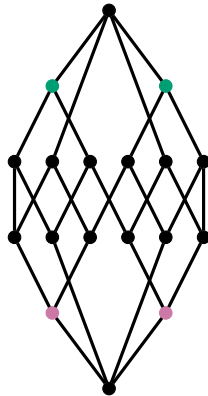
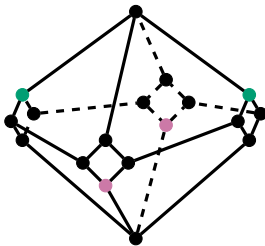
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Cubes

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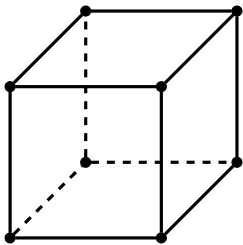
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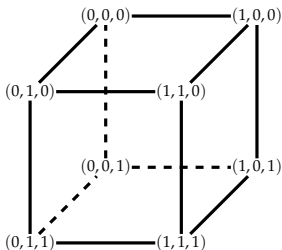
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Cubes

- **binary tuple**: integer tuple (u_1, u_2, \dots, u_n) such that
 - $u_i \in \{0, 1\}$ $\rightsquigarrow \text{Bin}(n)$



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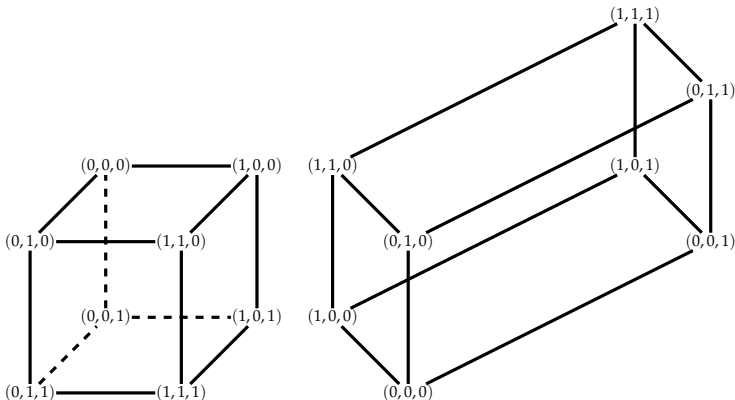
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- **binary tuple**: integer tuple (u_1, u_2, \dots, u_n) such that
 - $u_i \in \{0, 1\}$ $\rightsquigarrow \text{Bin}(n)$
- **Boolean lattice**:
 $\mathbf{Bool}(n) \stackrel{\text{def}}{=} (\text{Bin}(n), \leq_{\text{comp}})$



Freehedra

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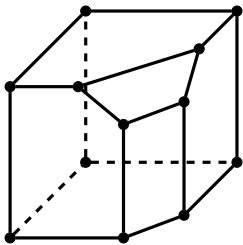
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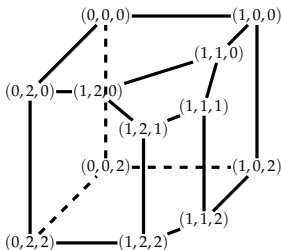
• **triword**: integer tuple (u_1, u_2, \dots, u_n) such that

• $u_i \in \{0, 1, 2\}$

• $u_1 \neq 2$

• $u_i = 0$ implies $u_j \neq 1$ for all $j > i$

$\rightsquigarrow \text{Tri}(n)$



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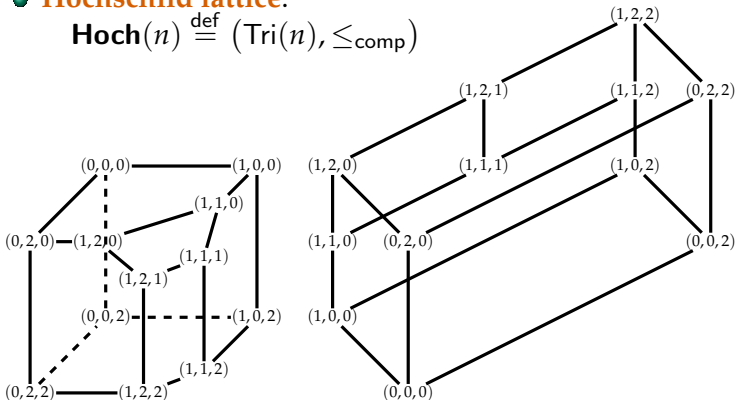
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 - $u_i \in \{0, 1, 2\}$
 - $u_1 \neq 2$
 - $u_i = 0$ implies $u_j \neq 1$ for all $j > i$ $\rightsquigarrow \text{Tri}(n)$
- **Hochschild lattice**:

$$\text{Hoch}(n) \stackrel{\text{def}}{=} (\text{Tri}(n), \leq_{\text{comp}})$$



Associahedra

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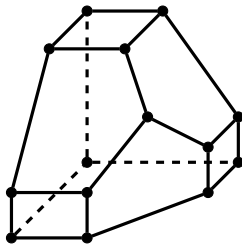
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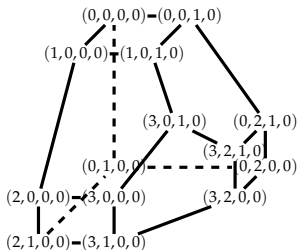
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- **bracket vector**: integer tuple (u_1, u_2, \dots, u_n) such that
 - $u_i \in \{0, 1, \dots, n - i\}$ $\rightsquigarrow \text{Brac}(n)$
 - $u_{i+j} \leq u_i - j$ for all $j \in \{0, 1, \dots, u_i\}$



Associahedra

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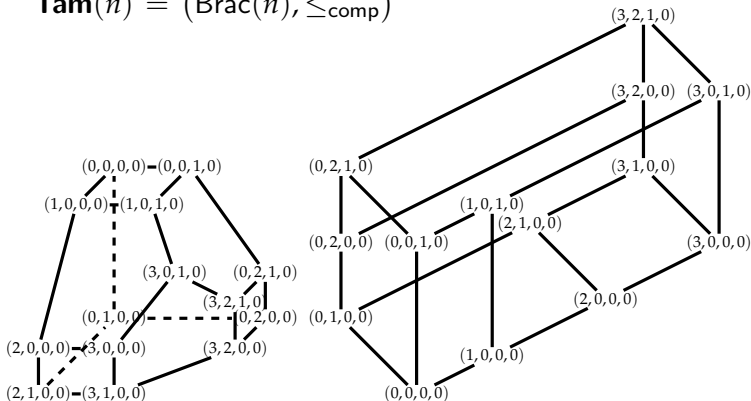
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 - $u_{i+j} \leq u_i - j$ for all $j \in \{0, 1, \dots, u_i\}$
- **Tamari lattice:**
 $\text{Tam}(n) \stackrel{\text{def}}{=} (\text{Brac}(n), \leq_{\text{comp}})$



More Examples

References

- permutohedron \rightsquigarrow weak order [Folklore]
- Coxeter-associahedron \rightsquigarrow Cambrian lattices [Reading, 2006]
- triangulations of cyclic polytope \rightsquigarrow higher Stasheff–Tamari posets [Edelman, Reiner, 1996]
- Stokes polytopes \rightsquigarrow Stokes posets [Baryshnikov, 2001], [Chapoton, 2016]
- non-kissing complex \rightsquigarrow Grid–Tamari order [Pylyavskyy, Petersen, Speyer, 2010], [McConville, 2017]
- Grassmann associahedron \rightsquigarrow Grassmann–Tamari order [Santos, Stump, Welker, 2017]
- graph associahedra \rightsquigarrow posets of tubings [Barnard, McConville, 2021]

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Faces of Polytopes

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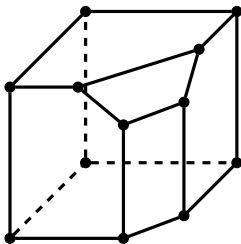
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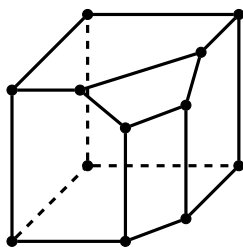
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- $\mathcal{P} \subseteq \mathbb{R}^d$.. polytope



Faces of Polytopes

- $\mathcal{P} \subseteq \mathbb{R}^d$.. polytope
- **face numbers:** $f_k \stackrel{\text{def}}{=} |\{A \in \mathcal{P} \mid \dim(A) = k\}|$



$$\begin{aligned}f_3 &= 1 \\f_2 &= 8 \\f_1 &= 18 \\f_0 &= 12 \\f_{-1} &= 1\end{aligned}$$

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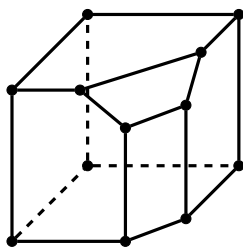
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- **face numbers:** $f_k \stackrel{\text{def}}{=} |\{A \in \mathcal{P} \mid \dim(A) = k\}|$
- **f -polynomial:** $f_{\mathcal{P}}(x) \stackrel{\text{def}}{=} \sum_{k=0}^d f_{k-1} x^{d-k}$



$$\begin{aligned}f_3 &= 1 \\f_2 &= 8 \\f_1 &= 18 \\f_0 &= 12 \\f_{-1} &= 1\end{aligned}$$

$$f_{\mathcal{P}}(x) = x^3 + 12x^2 + 18x + 8$$

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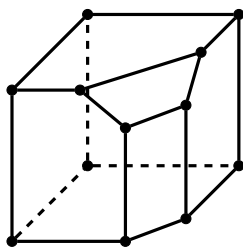
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- **\tilde{f} -polynomial:** $\tilde{f}_{\mathcal{P}}(x) \stackrel{\text{def}}{=} \sum_{k=0}^d f_{k-1} x^k$



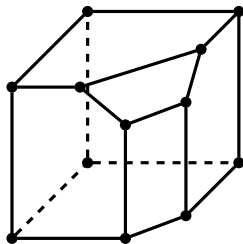
$$\begin{aligned}f_3 &= 1 \\f_2 &= 8 \\f_1 &= 18 \\f_0 &= 12 \\f_{-1} &= 1\end{aligned}$$

$$\tilde{f}_{\mathcal{P}}(x) = 8x^3 + 18x^2 + 12x + 1$$

Faces of Polytopes

- $\mathcal{P} \subseteq \mathbb{R}^d$.. polytope

- **h -numbers**: $h_k \stackrel{\text{def}}{=} \sum_{i=0}^k (-1)^{k-i} \binom{d-i}{d-k} f_{i-1}$



$$\begin{aligned}h_3 &= 3 \\h_2 &= -3 \\h_1 &= 9 \\h_0 &= 1\end{aligned}$$

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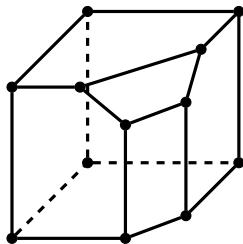
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- **h -polynomial:** $h_{\mathcal{P}}(x) \stackrel{\text{def}}{=} \sum_{k=0}^d h_k x^{d-k}$



$$\begin{aligned}h_3 &= 3 \\h_2 &= -3 \\h_1 &= 9 \\h_0 &= 1\end{aligned}$$

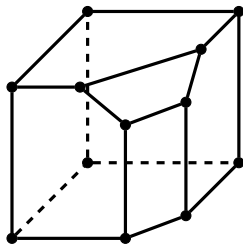
$$h_{\mathcal{P}}(x) = x^3 + 9x^2 - 3x + 3$$

Faces of Polytopes

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- **h -numbers:** $h_k \stackrel{\text{def}}{=} \sum_{i=0}^k (-1)^{k-i} \binom{d-i}{d-k} f_{i-1}$

- **h -polynomial:** $h_{\mathcal{P}}(x) = f_{\mathcal{P}}(x-1)$

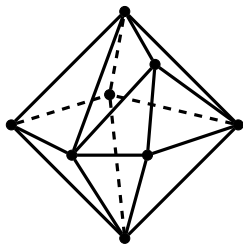


$$\begin{aligned}f_3 &= 1 \\f_2 &= 8 \\f_1 &= 18 \\f_0 &= 12 \\f_{-1} &= 1\end{aligned}$$

$$h_{\mathcal{P}}(x) = (x-1)^3 + 12(x-1)^2 + 18(x-1) + 8$$

Refined Face Enumeration

- $\mathcal{P} \subseteq \mathbb{R}^d$.. polytope



$$f_3 = 1$$

$$f_2 = 12$$

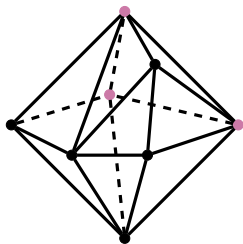
$$f_1 = 18$$

$$f_0 = 8$$

$$f_{-1} = 1$$

Refined Face Enumeration

- $\mathcal{P} \subseteq \mathbb{R}^d$.. polytope; fix a set M of vertices



$$f_3 = 1$$

$$f_2 = 12$$

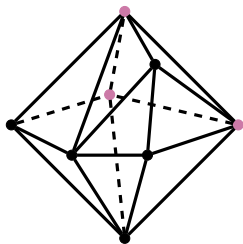
$$f_1 = 18$$

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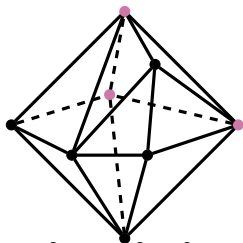
- $\mathcal{P} \subseteq \mathbb{R}^d$.. polytope; fix a set M of vertices
- for a face $A \in \mathcal{P}$ define:
 - $\text{pos}(A) \stackrel{\text{def}}{=} |A \setminus M|$
 - $\text{neg}(A) \stackrel{\text{def}}{=} |A \cap M|$
- **F-triangle:** $F_{\mathcal{P}}(x, y) \stackrel{\text{def}}{=} \sum_{A \in \mathcal{P}} x^{\text{pos}(A)} y^{\text{neg}(A)}$



$$\begin{aligned}f_3 &= 1 \\f_2 &= 12 \\f_1 &= 18 \\f_0 &= 8 \\f_{-1} &= 1\end{aligned}$$

Refined Face Enumeration

- $\mathcal{P} \subseteq \mathbb{R}^d$.. polytope; fix a set M of vertices
- for a face $A \in \mathcal{P}$ define:
 - $\text{pos}(A) \stackrel{\text{def}}{=} |A \setminus M|$
 - $\text{neg}(A) \stackrel{\text{def}}{=} |A \cap M|$
- **F-triangle:** $F_{\mathcal{P}}(x, y) \stackrel{\text{def}}{=} \sum_{A \in \mathcal{P}} x^{\text{pos}(A)} y^{\text{neg}(A)}$



$$\begin{aligned}f_3 &= 1 \\f_2 &= 12 \\f_1 &= 18 \\f_0 &= 8 \\f_{-1} &= 1\end{aligned}$$

$$F_{\mathcal{P}}(x, y) = 3x^3 + 5x^2y + 3xy^2 + y^3 + 7x^2 + 8xy + 3y^2 + 5x + 3y + 1$$

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Roots

- ε_i .. i^{th} unit vector in \mathbb{R}^{d+1}
- **(positive) roots:** $\alpha_{i,j} \stackrel{\text{def}}{=} \varepsilon_i - \varepsilon_j$ for $i < j$ $\rightsquigarrow \Phi_+(d)$
- **simple roots:** $\alpha_{i,i+1}$ $\rightsquigarrow \Pi(d)$

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The $F=H$ -
Correspondence

Roots

- ε_i .. i^{th} unit vector in \mathbb{R}^{d+1}
- **(positive) roots:** $\alpha_{i,j} \stackrel{\text{def}}{=} \varepsilon_i - \varepsilon_j$ for $i < j$ $\rightsquigarrow \Phi_+(d)$
- **simple roots:** $\alpha_{i,i+1}$ $\rightsquigarrow \Pi(d)$
- **root system:** $\Phi(d) \stackrel{\text{def}}{=} \{\pm\alpha_{i,j} \mid 1 \leq i < j \leq d+1\}$

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Chapoton's F -
and
 H -Triangle

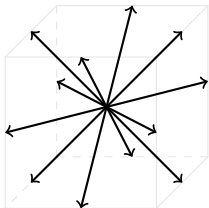
ν -
Associahedra

The $F=H$ -
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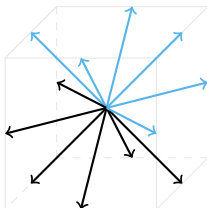
$d = 3$



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$d = 3$



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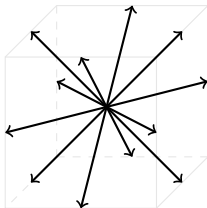
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The $F=H$ -
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- **almost positive roots:** $\Phi_{\geq -1}(d) \stackrel{\text{def}}{=} \Phi_+(d) \uplus -\Pi(d)$

$d = 3$



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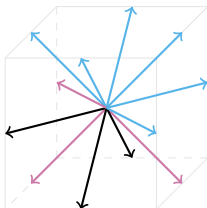
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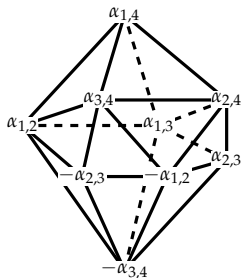
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$$\tilde{f}_{\text{Clus}(3)}(x) = 14x^3 + 21x^2 + 9x + 1$$

$$h_{\text{Clus}(3)}(x) = x^3 + 6x^2 + 6x + 1$$

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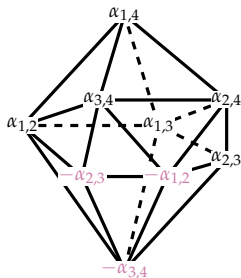
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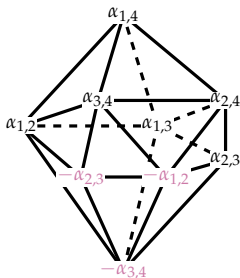
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The $F=H$ -
Correspondence

- for $A \in \text{Clus}(d)$ let
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 - $\text{pos}(A) \stackrel{\text{def}}{=} |A \setminus (-\Pi(d))|$
- **F -triangle:** $F_d(x, y) \stackrel{\text{def}}{=} \sum_{A \in \text{Clus}(d)} x^{\text{pos}(A)} y^{\text{neg}(A)}$

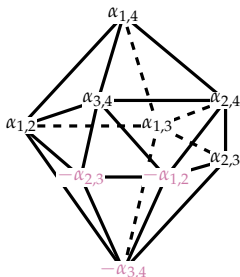
Chapoton's F -Triangle

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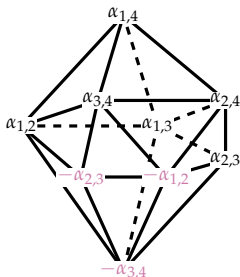
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$$F_3(x, y) = 5x^3 + 5x^2y + 3xy^2 + y^3 + 10x^2 + 8xy + 3y^2 + 6x + 3y + 1$$

Chapoton's F -Triangle

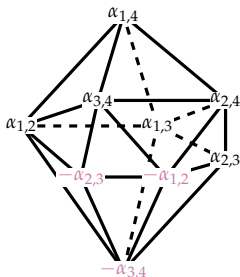
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Chapoton's F -Triangle

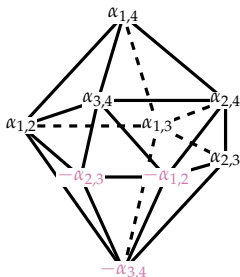
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$$F_3(x, x) = 14x^3 + 21x^2 + 9x + 1 = \tilde{f}_{\text{Clus}(3)}(x)$$

Chapoton's F -Triangle

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Chapoton's H -Triangle

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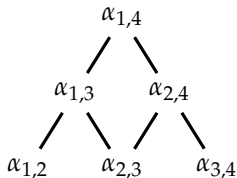
ν -
Associahedra

The $F=H$ -
Correspondence

- **root order:** $\alpha \preceq \beta$ if and only if $\beta - \alpha \in \text{Span}_{\mathbb{N}} \Pi(d)$

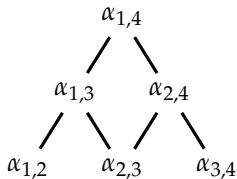
Chapoton's H -Triangle

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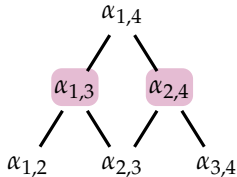
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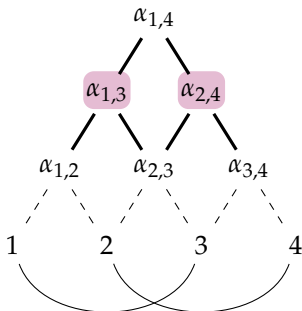
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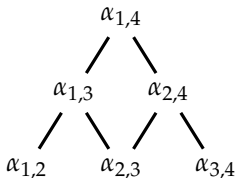
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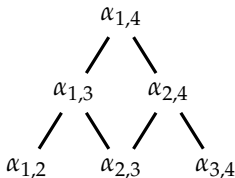
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Chapoton's H -Triangle

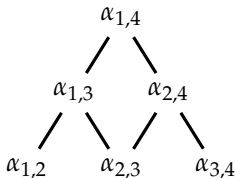
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$$H_3(x, y) = x^3 y^3 + 3x^2 y^2 + 2x^2 y + x^2 + 3xy + 3x + 1$$

Chapoton's H -Triangle

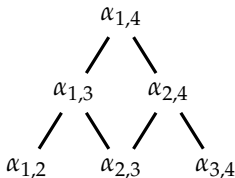
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$$H_3(x, 1) = x^3 + 6x^2 + 6x + 1$$

Chapoton's H -Triangle

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$$H_3(x, 1) = x^3 + 6x^2 + 6x + 1 = h_{\text{Clus}(3)}(x)$$

The $F=H$ -Correspondence

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Conjecture (F. Chapoton, 2006)

For $d \geq 1$,

$$F_d(x, y) = x^d H_d \left(\frac{x+1}{x}, \frac{y+1}{x+1} \right).$$

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$$\begin{aligned} h_{\text{Clus}(d)}(x) &= f_{\text{Clus}(d)}(x-1) \\ &= (x-1)^d \tilde{f}_{\text{Clus}(d)} \left(\frac{1}{x-1} \right) \\ &= (x-1)^d F_d \left(\frac{1}{x-1}, \frac{1}{x-1} \right) \\ &= H_d(x, 1) \end{aligned}$$

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- F. Chapoton: $\frac{\partial}{\partial y} F_d(x, y) = \sum_{\alpha \in \Pi(d)} F_{d \setminus \alpha}(x, y)$
- M. Thiel: $\frac{\partial}{\partial y} H_d(x, y) = x \sum_{\alpha \in \Pi(d)} H_{d \setminus \alpha}(x, y)$

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Theorem (M. Thiel, 2014)

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The $F=H$ -
Correspondence

- **Dyck path:** northeast path from $(0,0)$ to (n,n) weakly above the diagonal $\rightsquigarrow \text{Dyck}(n)$

Dyck Paths

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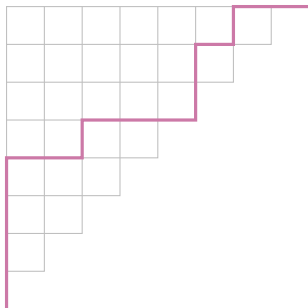
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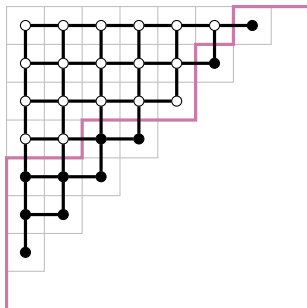
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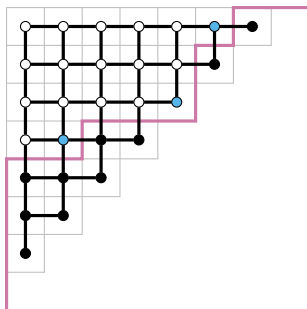
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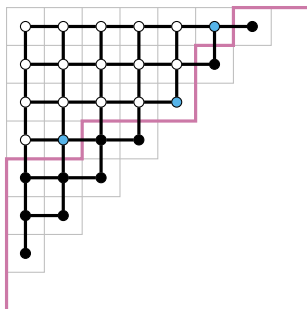
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- $A \in \text{Nonn}(n)$ corresponds to **valleys** of $\mu \in \text{Dyck}(n)$
- simple roots contained in A correspond to **returns**



Dyck Paths

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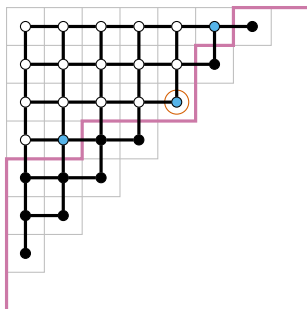
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Corollary

For $n \geq 1$,

$$H_n(x, y) = \sum_{\mu \in \text{Dyck}(n)} x^{\text{val}(\mu)} y^{\text{ret}(\mu)}.$$

The Tamari Lattice

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Posets from
Polytopes

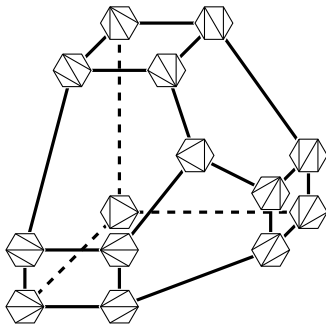
Refined Face
Enumeration

Chapoton's F -
and
 H -Triangle

ν -
Associahedra

The $F=H$ -
Correspondence

- **(simple) associahedron**: triangulations of a $n+3$ -gon connected by diagonal flips $\rightsquigarrow \text{Asso}(n)$
- $\text{Clus}(n)$ is the dual polytope of $\text{Asso}(n)$



The Tamari Lattice

Posets from
Polytopes and
Refined Face
Enumeration

Henri Mühle

Posets from
Polytopes

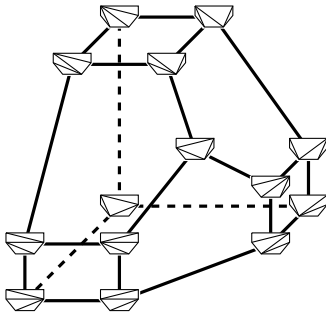
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The $F=H$ -
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- orient edges of $\text{Asso}(n)$ according to slope



The Tamari Lattice

Posets from
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Posets from
Polytopes

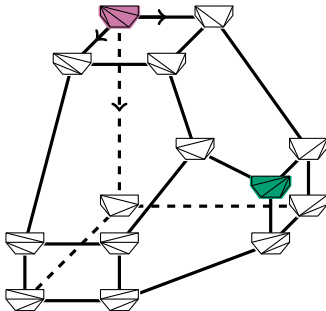
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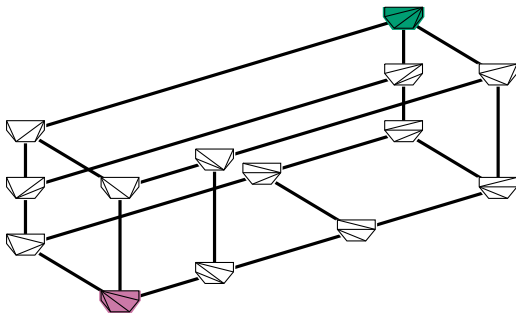
The $F=H$ -
Correspondence

- orient edges of $\text{Asso}(n)$ according to slope



The Tamari Lattice

- orient edges of $\text{Asso}(n)$ according to slope $\rightsquigarrow \mathbf{Tam}(n)$



Posets from
Polytopes and
Refined Face
Enumeration

Henri Mühle

Posets from
Polytopes

Refined Face
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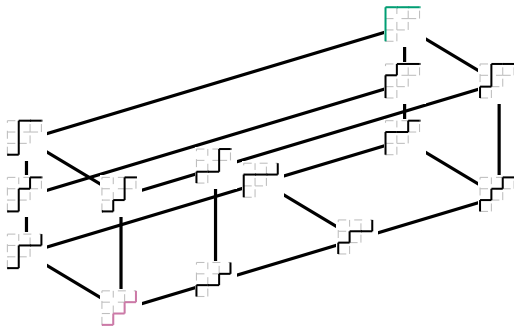
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The $F=H$ -
Correspondence

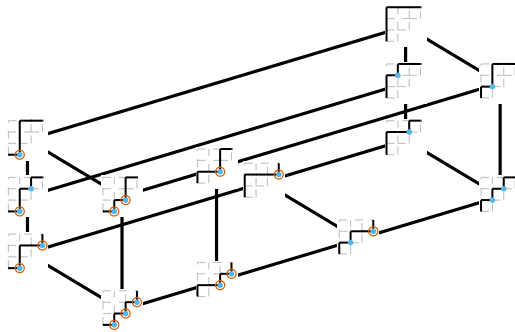
The Tamari Lattice

- orient edges of $\text{Asso}(n)$ according to slope $\rightsquigarrow \mathbf{Tam}(n)$



The Tamari Lattice

- orient edges of $\text{Asso}(n)$ according to slope $\rightsquigarrow \mathbf{Tam}(n)$



Posets from
Polytopes and
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Posets from
Polytopes

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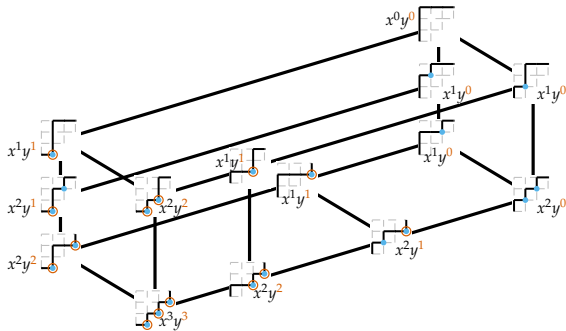
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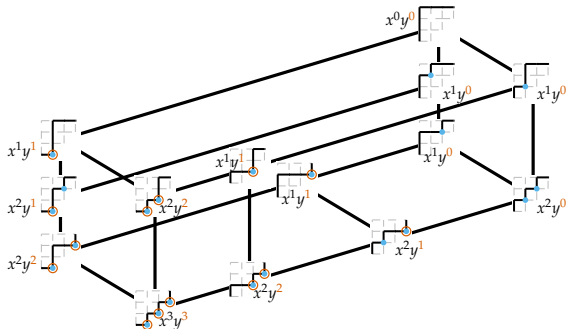
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The Tamari Lattice

- orient edges of $\text{Asso}(n)$ according to slope $\rightsquigarrow \mathbf{Tam}(n)$



$$H_3(x, y) = x^3y^3 + 3x^2y^2 + 2x^2y + x^2 + 3xy + 3x + 1$$

The ν -Tamari Lattice via Paths

- fix a northeast path ν

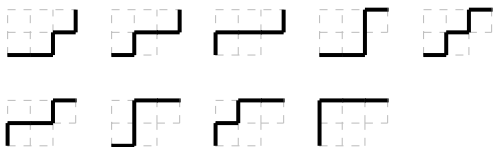
$$\nu = EENEN$$



The ν -Tamari Lattice via Paths

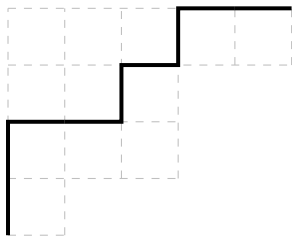
- fix a northeast path ν
- ν -path: northeast paths weakly above $\nu \rightsquigarrow \text{Dyck}(\nu)$

$\nu = EENEN$



The ν -Tamari Lattice via Paths

- **rotating** ν -paths by valleys



Posets from
Polytopes and
Refined Face
Enumeration

Henri Mühle

Posets from
Polytopes

Refined Face
Enumeration

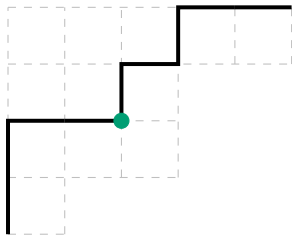
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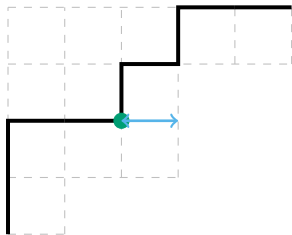
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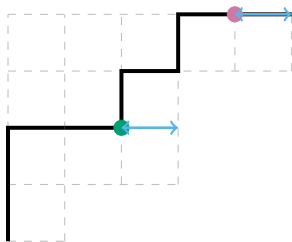
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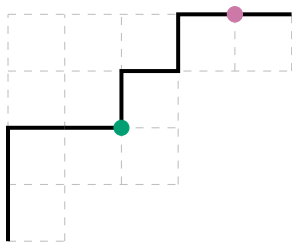
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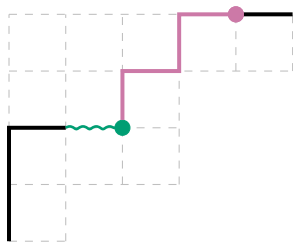
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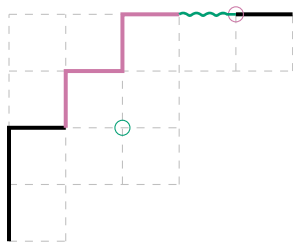
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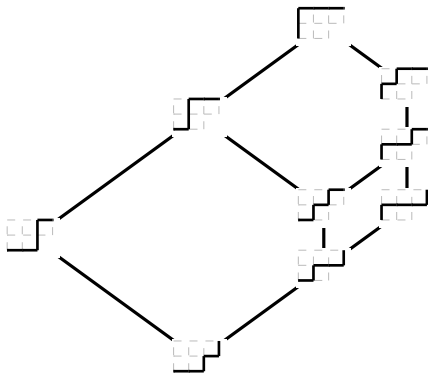
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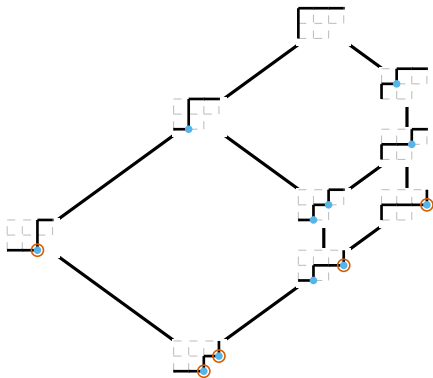
The ν -Tamari Lattice via Paths

- fix a northeast path ν
- ν -**path**: northeast paths weakly above ν $\rightsquigarrow \text{Dyck}(\nu)$
- ν -**Tamari lattice**: rotation order on $\text{Dyck}(\nu)$ $\rightsquigarrow \mathbf{Tam}(\nu)$



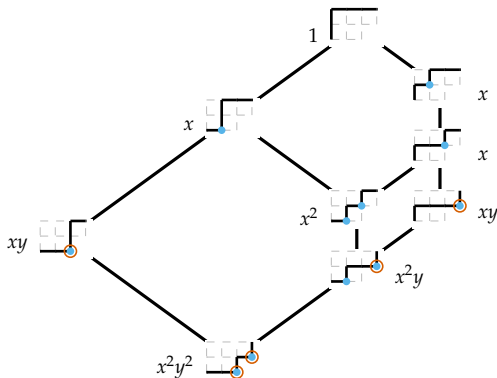
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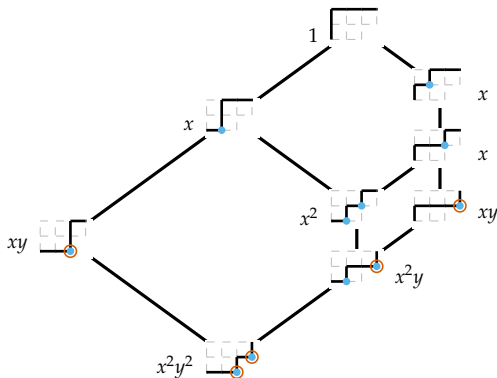
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$$H_{EENEN}(x, y) = x^2y^2 + x^2y + x^2 + 2xy + 3x + 1$$

ν -Trees

Posets from
Polytopes and
Refined Face
Enumeration

Henri Mühle

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Polytopes

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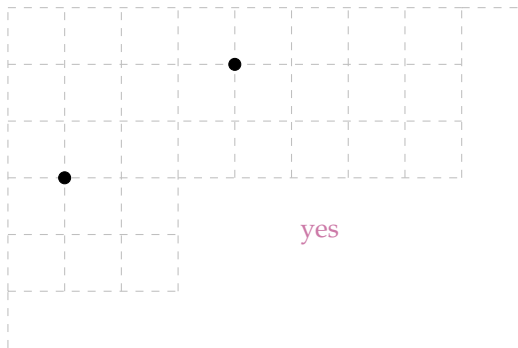
The $F=H$ -
Correspondence

- ν -incompatible nodes



ν -Trees

- ν -incompatible nodes



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Posets from
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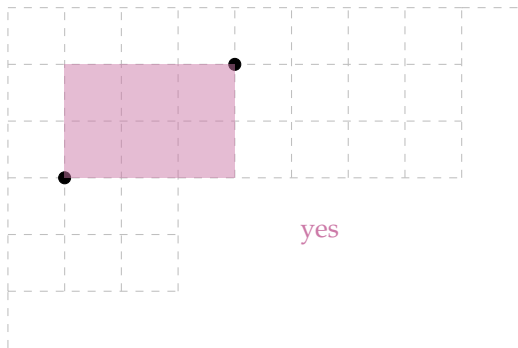
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The $F=H$ -
Correspondence

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- ν -incompatible nodes



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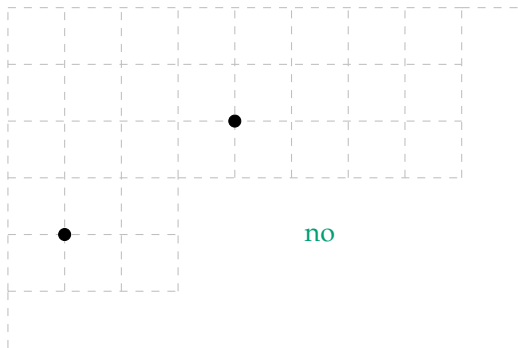
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Posets from
Polytopes

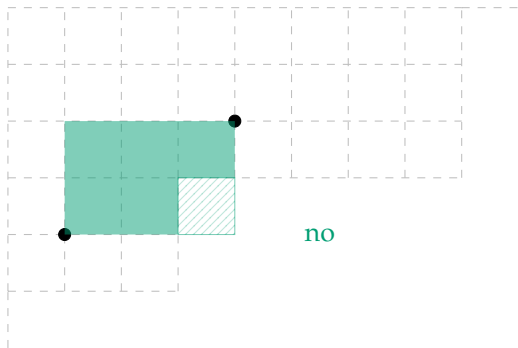
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ν -Trees

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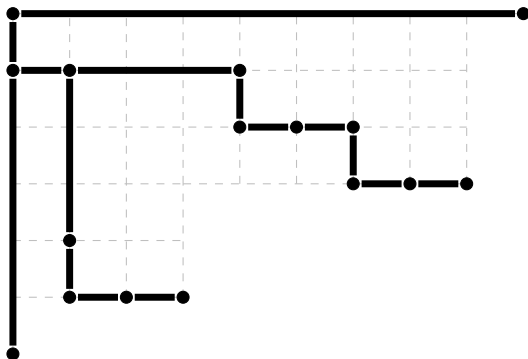
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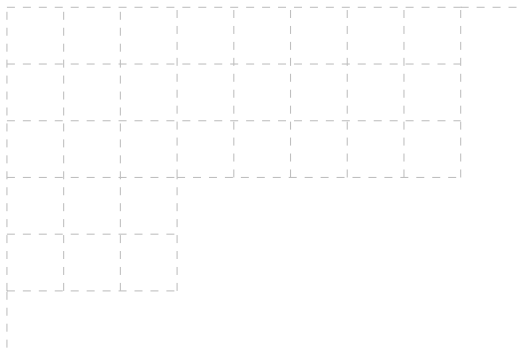
- ν -incompatible nodes
- ν -tree: maximal collection of ν -compatible nodes

$\rightsquigarrow \text{Tree}(\nu)$



From ν -Paths to ν -Trees

- the **right-flushing** bijection



Posets from
Polytopes and
Refined Face
Enumeration

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Posets from
Polytopes

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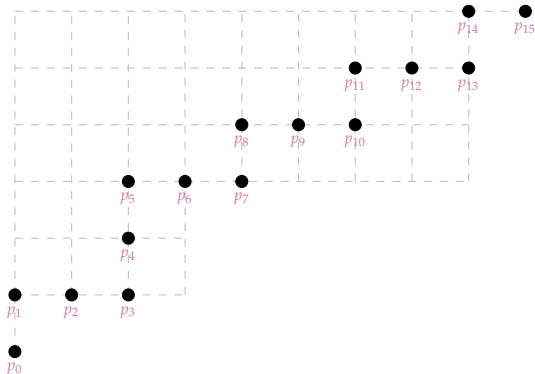
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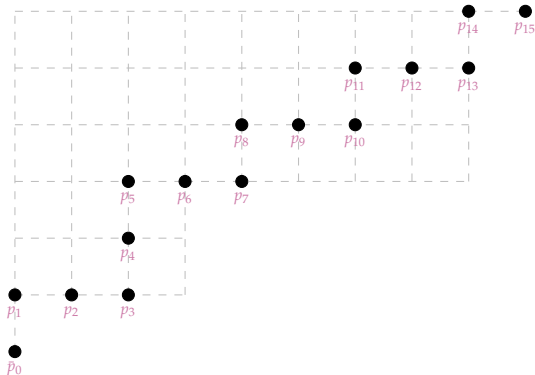
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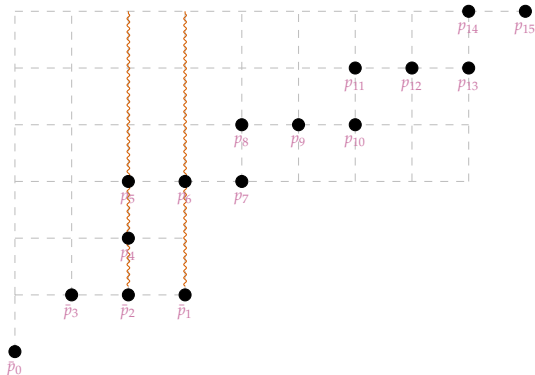
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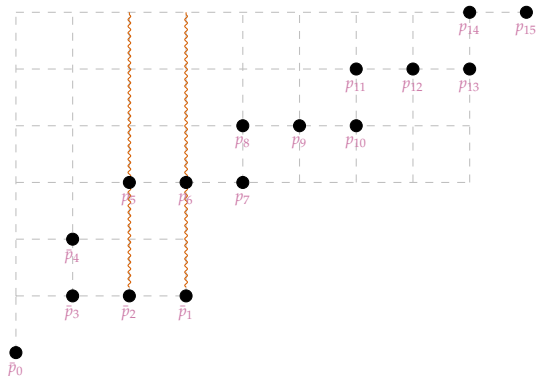
From ν -Paths to ν -Trees

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From ν -Paths to ν -Trees

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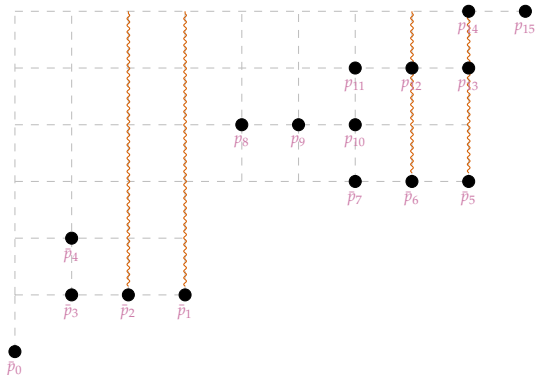
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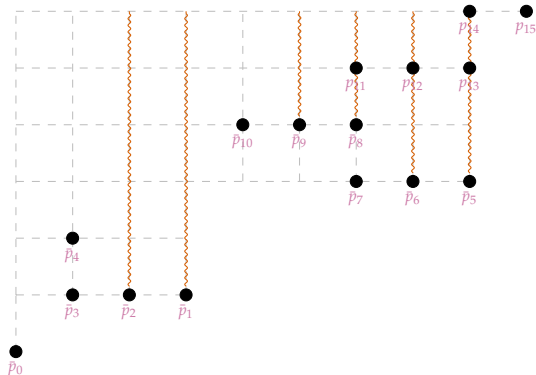
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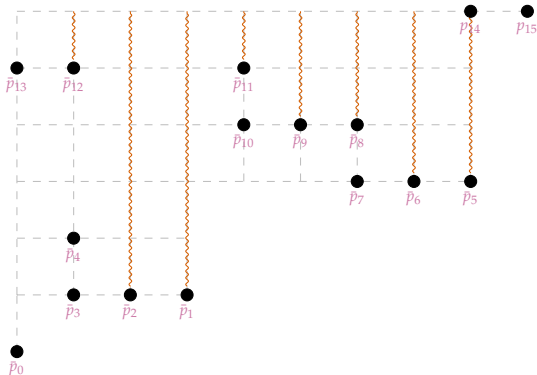
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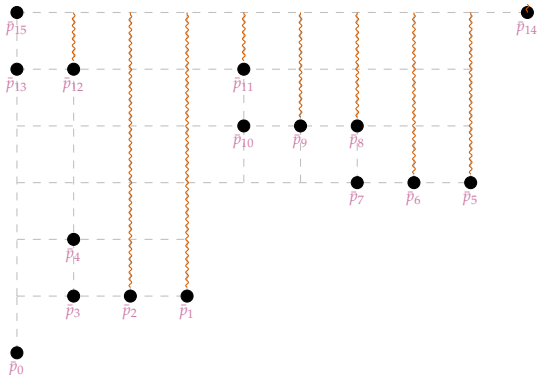
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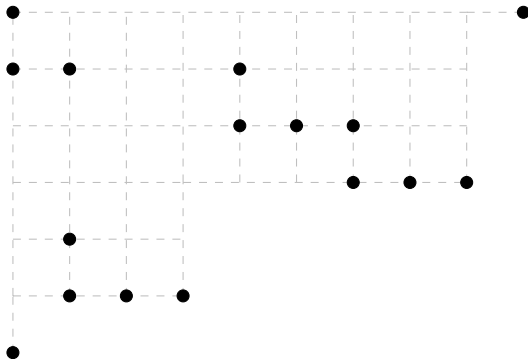
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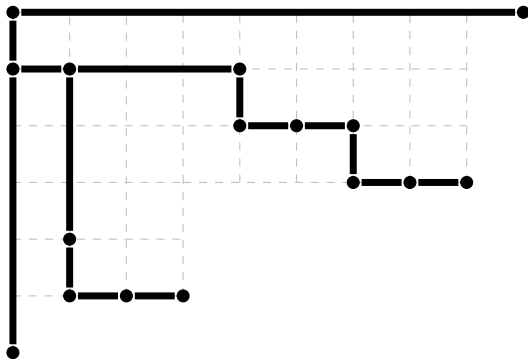
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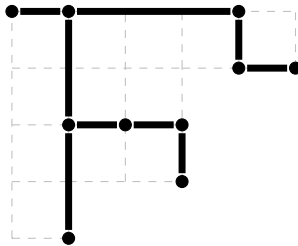
- the **right-flushing** bijection

Theorem (C. Ceballos, A. Padrol & C. Sarmiento, 2020)

Right-flushing is a bijection from $\text{Dyck}(\nu)$ to $\text{Tree}(\nu)$.

The ν -Tamari Lattice via Trees

- **rotating** ν -trees by ascent nodes



Posets from
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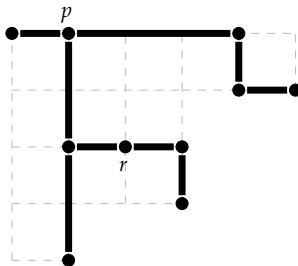
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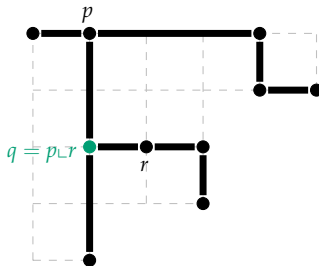
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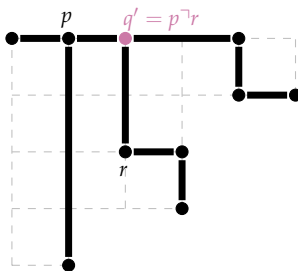
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The ν -Tamari Lattice via Trees

Theorem (C. Ceballos, A. Padrol & C. Sarmiento, 2020)

Right-flushing converts rotation on ν -paths into rotation on ν -trees.

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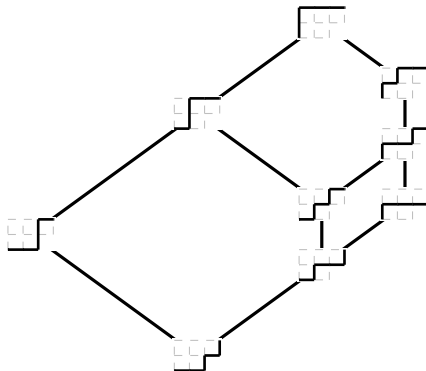
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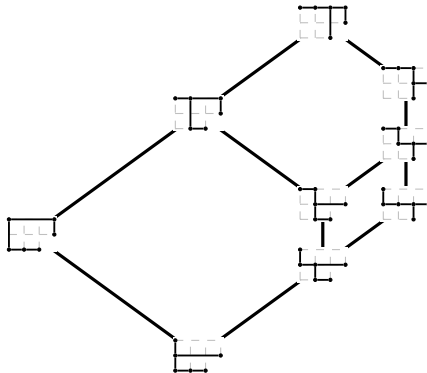
ν -
Associahedra

The $F=H$ -
Correspondence

The ν -Tamari Lattice via Trees

Theorem (C. Ceballos, A. Padrol & C. Sarmiento, 2020)

Right-flushing converts rotation on ν -paths into rotation on ν -trees.



The ν -Associahedron

- ν -face: collection of pairwise ν -compatible nodes
- ν -Tamari complex: simplicial complex of ν -faces

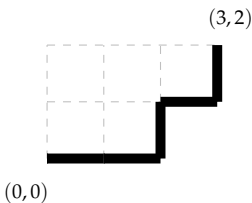
$$\rightsquigarrow \mathcal{TC}(\nu)$$

The ν -Associahedron

- ν -face: collection of pairwise ν -compatible nodes
- ν -Tamari complex: simplicial complex of ν -faces

$\rightsquigarrow \mathcal{TC}(\nu)$

$\nu = EENEN$

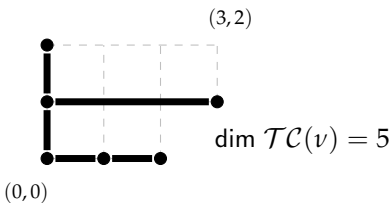


The ν -Associahedron

- ν -face: collection of pairwise ν -compatible nodes
- ν -Tamari complex: simplicial complex of ν -faces

$$\rightsquigarrow \mathcal{TC}(\nu)$$

$\nu = EENEN$



The ν -Associahedron

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Refined Face
Enumeration

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Posets from
Polytopes

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Chapoton's F -
and
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The $F=H$ -
Correspondence

- **covering ν -face**: ν -face containing top-left corner and at least one node per row and column

The ν -Associahedron

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Posets from
Polytopes

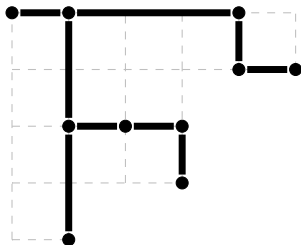
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The ν -Associahedron

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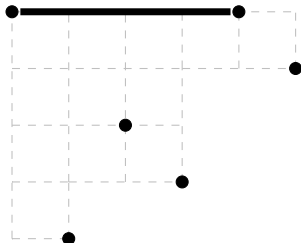
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The ν -Associahedron

- **covering ν -face**: ν -face containing top-left corner and at least one node per row and column
- **ν -associahedron**: polytopal complex of covering ν -faces $\rightsquigarrow \text{Asso}(\nu)$

The ν -Associahedron

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- **covering ν -face**: ν -face containing top-left corner and at least one node per row and column
- **ν -associahedron**: polytopal complex of covering ν -faces $\rightsquigarrow \text{Asso}(\nu)$

Theorem (C. Ceballos, A. Padrol & C. Sarmiento, 2019)

$\text{Asso}(\nu)$ is a polytopal complex dual to the complex of interior faces of $\mathcal{TC}(\nu)$.

The ν -Associahedron

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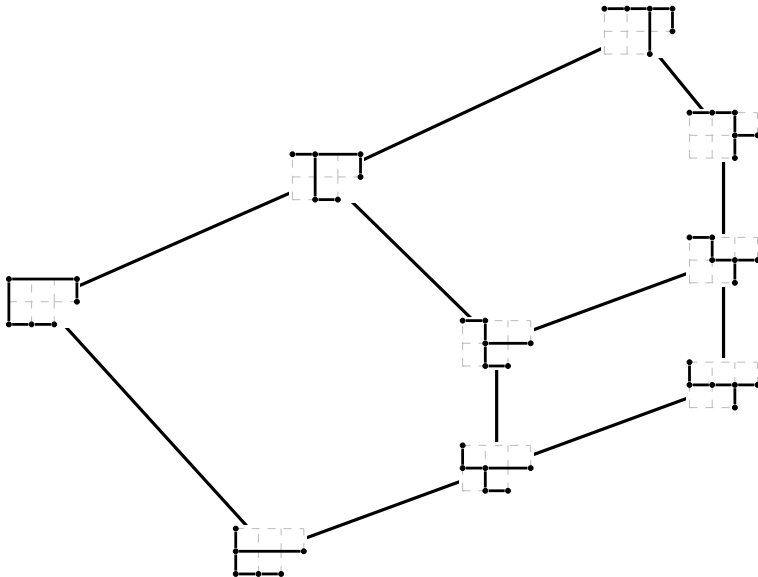
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The ν -Associahedron

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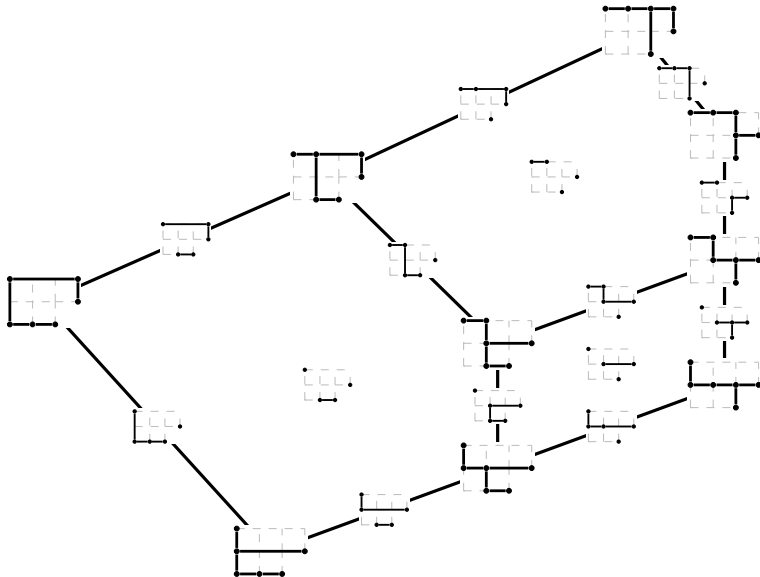
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Outline

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- 1 Posets from Polytopes
- 2 Refined Face Enumeration
- 3 Chapoton's F - and H -Triangle
- 4 ν -Associahedra
- 5 The $F=H$ -Correspondence

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- **relevant** position: coordinate in first column and in row of a valley of ν

Some Statistics

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Posets from
Polytopes

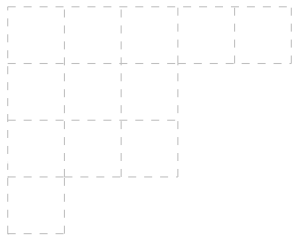
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Some Statistics

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Posets from
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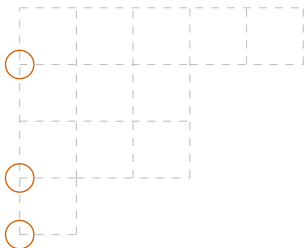
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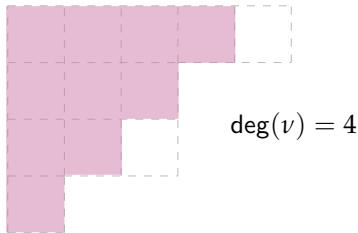
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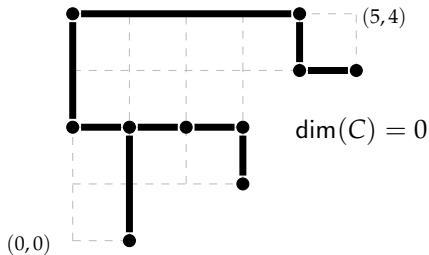
The $F=H$ -
Correspondence

- $\deg(\nu) \stackrel{\text{def}}{=} \max\{\text{val}(\mu) \mid \mu \in \text{Dyck}(\nu)\}$



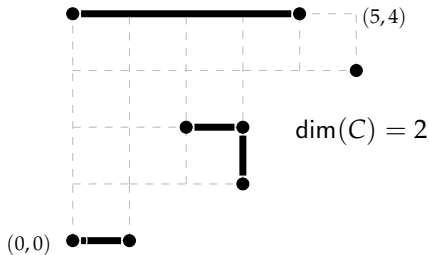
Some Statistics

- ν goes from $(0,0)$ to (m,n) ; $C \in \text{Asso}(\nu)$
- $\text{deg}(\nu) \stackrel{\text{def}}{=} \max\{\text{val}(\mu) \mid \mu \in \text{Dyck}(\nu)\}$
- $\text{dim}(C) \stackrel{\text{def}}{=} m + n + 1 - |C|$



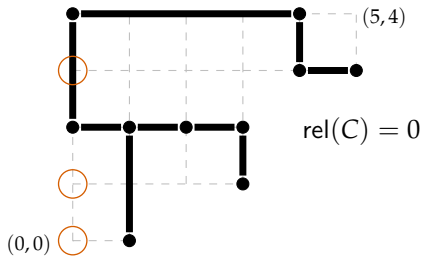
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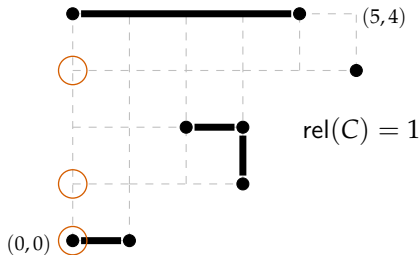
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- $\text{rel}(C)$ is number of relevant positions filled by C



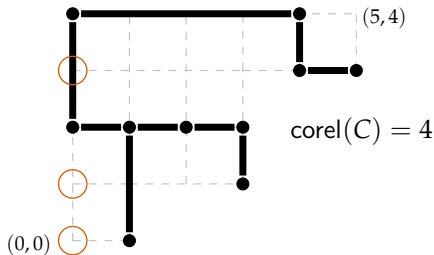
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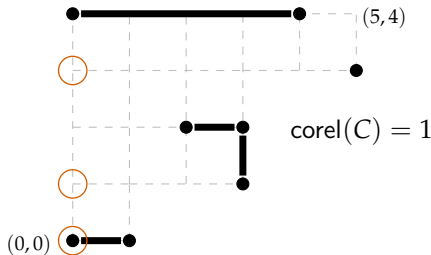
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- $\text{dim}(C) \stackrel{\text{def}}{=} m + n + 1 - |C|$
- $\text{rel}(C)$ is number of relevant positions filled by C
- $\text{corel}(C) \stackrel{\text{def}}{=} \text{deg}(\nu) - \text{dim}(C) - \text{rel}(C)$



Some Statistics

- ν goes from $(0,0)$ to (m,n) ; $C \in \text{Asso}(\nu)$
- $\text{deg}(\nu) \stackrel{\text{def}}{=} \max\{\text{val}(\mu) \mid \mu \in \text{Dyck}(\nu)\}$
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Some Statistics

- $C \in \text{Asso}(v)$
- $\text{rel}(C)$ is number of relevant positions filled by C
- $\text{asc}(C)$ is number of ascent nodes of C

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Correspondence

Some Statistics

- $C \in \text{Asso}(v)$
- $\text{rel}(C)$ is number of relevant positions filled by C
- $\text{asc}(C)$ is number of ascent nodes of C

Lemma (C. Ceballos & , 2020)

Right-flushing sends val to asc and ret to rel.

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- recall: $H_\nu(x, y) = \sum_{\mu \in \text{Dyck}(\nu)} x^{\text{val}(\mu)} y^{\text{ret}(\mu)}$

Corollary (C. Ceballos & , 2020)

For any northeast path ν ,

$$H_\nu(x, y) = \sum_{T \in \text{Tree}(\nu)} x^{\text{asc}(T)} y^{\text{rel}(T)}.$$

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recall:

$$F_d(x, y) = x^d H_d \left(\frac{x+1}{x}, \frac{y+1}{x+1} \right)$$

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$$F_\nu(x, y) = x^{\deg(\nu)} H_\nu \left(\frac{x+1}{x}, \frac{y+1}{x+1} \right)$$

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$$\begin{aligned} F_\nu(x, y) &= x^{\deg(\nu)} H_\nu \left(\frac{x+1}{x}, \frac{y+1}{x+1} \right) \\ &= x^{\deg(\nu)} \sum_{T \in \overline{\text{Tree}}(\nu)} \left(\frac{x+1}{x} \right)^{\text{asc}(T)} \left(\frac{y+1}{x+1} \right)^{\text{rel}(T)} \end{aligned}$$

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$$\begin{aligned}F_{\nu}(x, y) &= x^{\deg(\nu)} H_{\nu} \left(\frac{x+1}{x}, \frac{y+1}{x+1} \right) \\&= x^{\deg(\nu)} \sum_{T \in \text{Tree}(\nu)} \left(\frac{x+1}{x} \right)^{\text{asc}(T)} \left(\frac{y+1}{x+1} \right)^{\text{rel}(T)} \\&= \sum_{T \in \text{Tree}(\nu)} x^{\deg(\nu) - \text{asc}(T)} (x+1)^{\text{asc}(T) - \text{rel}(T)} (y+1)^{\text{rel}(T)}\end{aligned}$$

The F -Triangle

- $\text{Asc}(T)$ is set of ascent nodes of T
- $\text{Rel}(T)$ is set of relevant nodes of T

$$F_v(x, y) = \sum_{T \in \text{Tree}(v)} x^{\deg(v) - \text{asc}(T)} (x + 1)^{\text{asc}(T) - \text{rel}(T)} (y + 1)^{\text{rel}(T)}$$

The F -Triangle

- $\text{Asc}(T)$ is set of ascent nodes of T
- $\text{Rel}(T)$ is set of relevant nodes of T

$$\begin{aligned} F_\nu(x, y) &= \sum_{T \in \text{Tree}(\nu)} x^{\deg(\nu) - \text{asc}(T)} (x + 1)^{\text{asc}(T) - \text{rel}(T)} (y + 1)^{\text{rel}(T)} \\ &= \sum_{T \in \text{Tree}(\nu)} \left(\sum_{A'' \subseteq \text{Asc}(T) \setminus \text{Rel}(T)} x^{\deg(\nu) - \text{rel}(T) - |A''|} \right. \\ &\quad \left. \times \sum_{A' \subseteq \text{Rel}(T)} y^{\text{rel}(T) - |A'|} \right) \end{aligned}$$

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Proposition (C. Ceballos & V. Pons, 2019)

The map $(T, A) \mapsto T \setminus A$ is a bijection from $\{(T, A) \mid T \in \text{Tree}(\nu), A \subseteq \text{Asc}(T)\}$ to $\text{Asso}(\nu)$.

$$\begin{aligned} F_\nu(x, y) &= \sum_{T \in \text{Tree}(\nu)} \left(\sum_{A'' \subseteq \text{Asc}(T) \setminus \text{Rel}(T)} x^{\deg(\nu) - \text{rel}(T) - |A''|} \right. \\ &\quad \left. \times \sum_{A' \subseteq \text{Rel}(T)} y^{\text{rel}(T) - |A'|} \right) \\ &= \sum_{T \in \text{Tree}(\nu)} \sum_{\substack{A \subseteq \text{Asc}(T) \\ A' = A \cap \text{Rel}(T) \\ A'' = A \setminus \text{Rel}(T)}} x^{\deg(\nu) - \text{rel}(T) - |A''|} y^{\text{rel}(T) - |A'|} \end{aligned}$$

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$$F_\nu(x, y) \stackrel{\text{def}}{=} \sum_{C \in \text{Asso}(\nu)} x^{\text{corel}(C)} y^{\text{rel}(C)}$$

The $F=H$ -Correspondence for ν -Associahedra

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Theorem (C. Ceballos & , 2021)

For every northeast path ν ,

$$F_\nu(x, y) = x^{\deg(\nu)} H_\nu \left(\frac{x+1}{x}, \frac{y+1}{x+1} \right).$$

$$\begin{aligned} F_\nu(x, y) &\stackrel{\text{def}}{=} \sum_{C \in \text{Asso}(\nu)} x^{\text{corel}(C)} y^{\text{rel}(C)} \\ &= \sum_{T \in \text{Tree}(\nu)} x^{\deg(\nu) - \text{asc}(T)} (x+1)^{\text{asc}(T) - \text{rel}(T)} (y+1)^{\text{rel}(T)} \end{aligned}$$

The $F=H$ -Correspondence for ν -Associahedra

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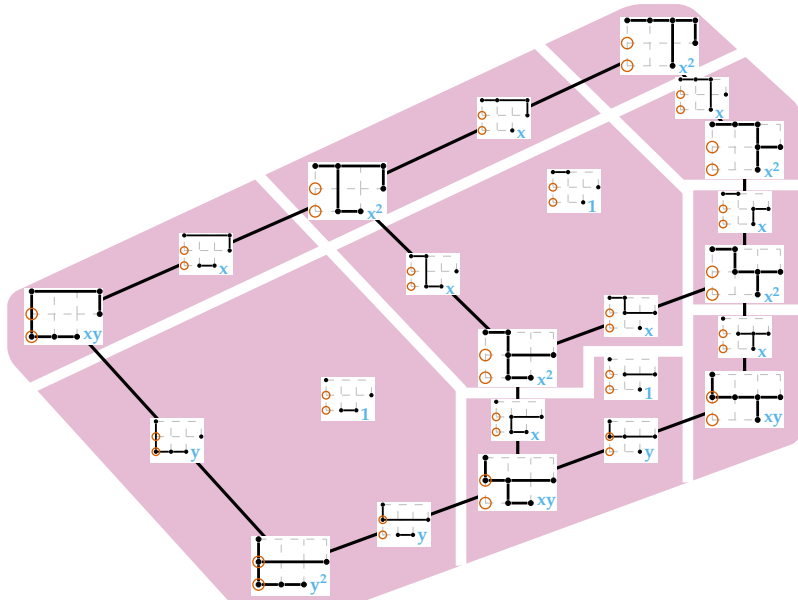
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The $F=H$ -Correspondence for ν -Associahedra

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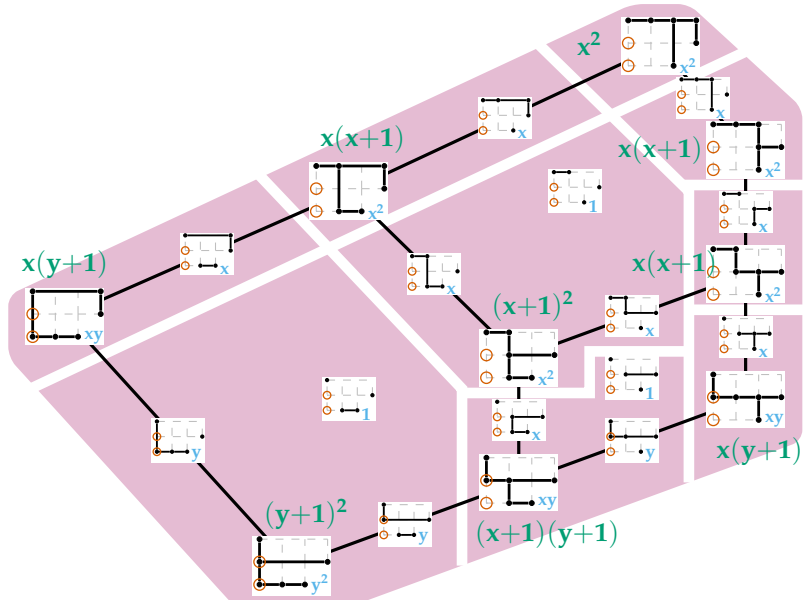
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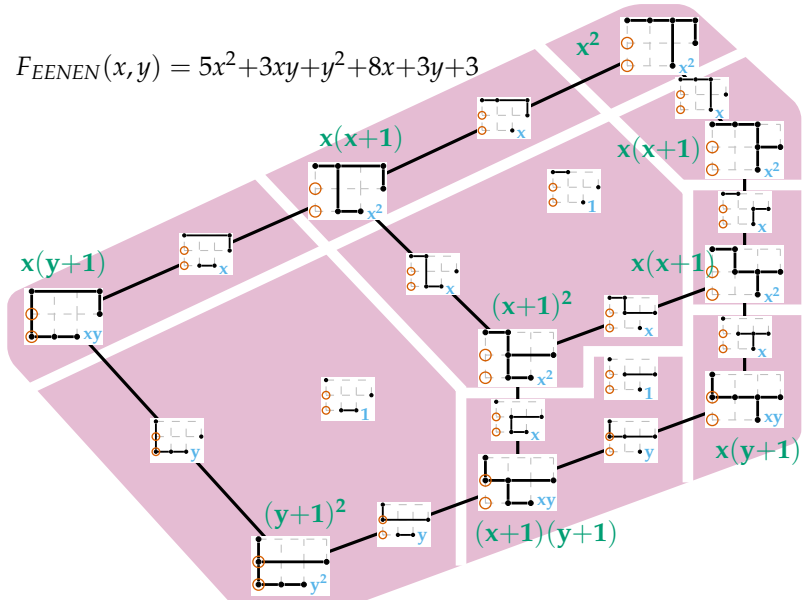
ν -
Associahedra

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The $F=H$ -Correspondence for ν -Associahedra

$$F_{EENEN}(x, y) = 5x^2 + 3xy + y^2 + 8x + 3y + 3$$



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F- and H-Triangles for Posets

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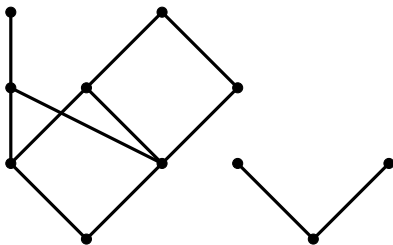
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and
H-Triangle

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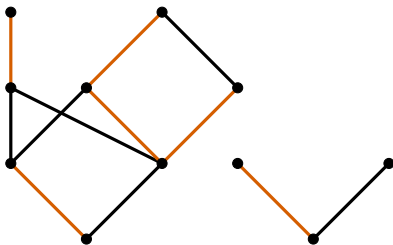
The *F*=*H*-
Correspondence

- $\mathbf{P} = (P, \leq); p \in P$
- $\text{Succ}(p) \stackrel{\text{def}}{=} \{p' \in P \mid p \triangleleft p'\}$
- $\text{out}(p) \stackrel{\text{def}}{=} |\text{Succ}(p)|$



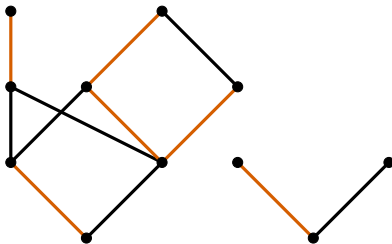
F - and H -Triangles for Posets

- $\mathbf{P} = (P, \leq); p \in P; \lambda \text{ .. } 01\text{-labeling}$
- $\text{Succ}(p) \stackrel{\text{def}}{=} \{p' \in P \mid p \triangleleft p'\}$
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F- and H-Triangles for Posets

- $\mathbf{P} = (P, \leq); p \in P; \lambda \text{ .. } 01\text{-labeling}$
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F- and H-Triangles for Posets

Posets from
Polytopes and
Refined Face
Enumeration

Henri Mühle

Posets from
Polytopes

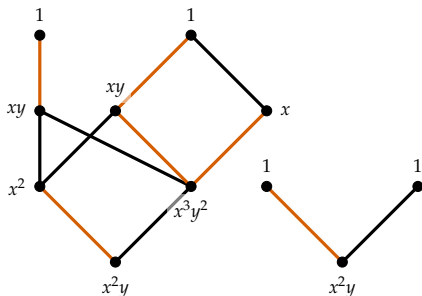
Refined Face
Enumeration

Chapoton's F-
and
H-Triangle

v-
Associahedra

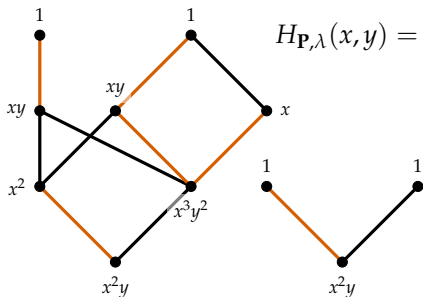
The F=H-
Correspondence

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F- and H-Triangles for Posets

- $\mathbf{P} = (P, \leq); p \in P; \lambda \text{ .. } 01\text{-labeling}$
- $H_{\mathbf{P},\lambda}(x, y) \stackrel{\text{def}}{=} \sum_{p \in P} x^{\text{out}(p)} y^{\text{mrk}(p)}$



$$H_{\mathbf{P},\lambda}(x, y) = x^3y^2 + 2x^2y + x^2 + 2xy + x + 4$$

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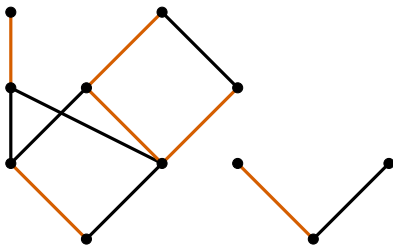
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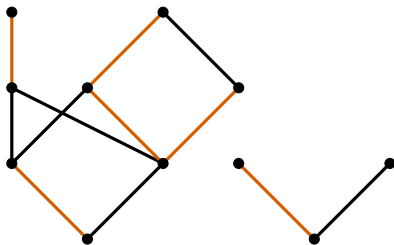
The $F=H$ -
Correspondence

- $\mathbf{P} = (P, \leq); p \in P$
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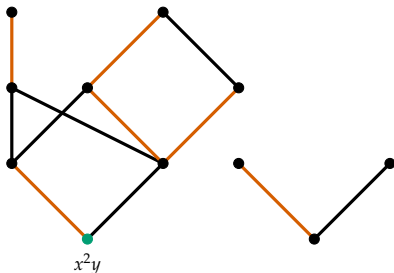
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- $\text{pos}(p, S) \stackrel{\text{def}}{=} \text{deg}(\mathbf{P}) - |S| - \text{neg}(p, S)$



F- and H-Triangles for Posets

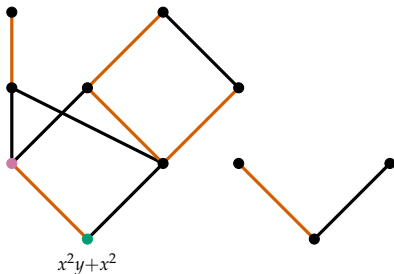
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x^2y

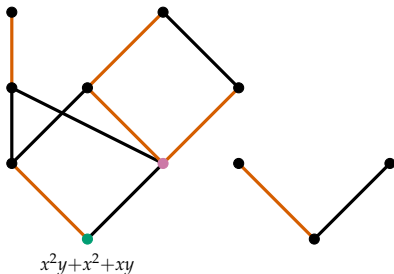
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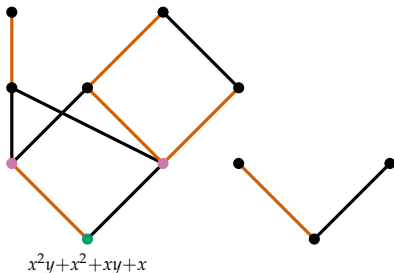
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Enumeration

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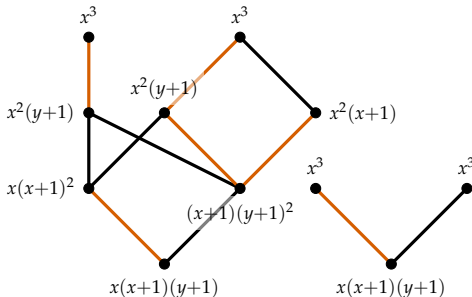
Chapoton's F-
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The F=H-
Correspondence

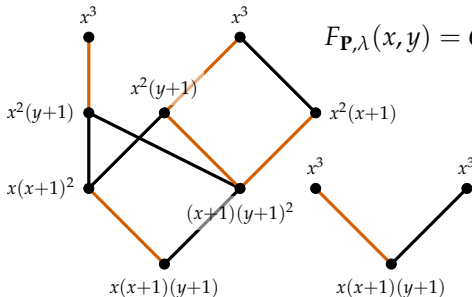
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F- and H-Triangles for Posets

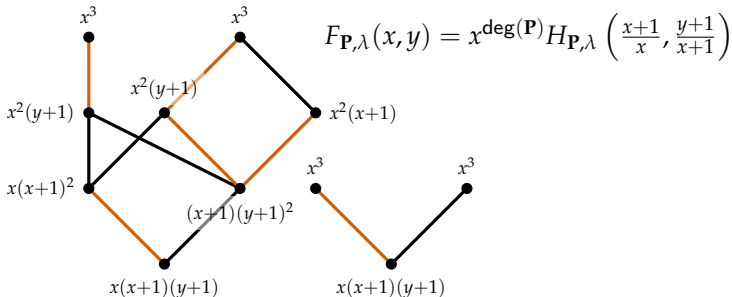
- $\mathbf{P} = (P, \leq); p \in P; S \subseteq \text{Succ}(p)$
- $F_{\mathbf{P}, \lambda}(x, y) \stackrel{\text{def}}{=} \sum_{p \in P} \sum_{S \subseteq \text{Succ}(p)} x^{\text{pos}(p, S)} y^{\text{neg}(p, S)}$



$$F_{\mathbf{P}, \lambda}(x, y) = 6x^3 + 4x^2y + xy^2 + 7x^2 + 4xy + y^2 + 4x + 2y + 1$$

F- and H-Triangles for Posets

- $\mathbf{P} = (P, \leq); p \in P; S \subseteq \text{Succ}(p)$
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The F=H-
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Theorem (C. Ceballos & , 2021)

For every finite poset \mathbf{P} and every 01-labeling λ ,

$$F_{\mathbf{P},\lambda}(x, y) = x^{\text{deg}(\mathbf{P})} H_{\mathbf{P},\lambda} \left(\frac{x+1}{x}, \frac{y+1}{x+1} \right).$$

Open Questions

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Enumeration

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The $F=H$ -
Correspondence

- can we find other (geometric) interpretations of the F - and the H -triangles?
- what is the relation between the two variants of the H -triangle?
- what is the relation with the M -triangle?
- we need more examples!

H -Triangle

M -Triangle

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Thank You.

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Examples

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Perspectivity

Posets from
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M-triangle

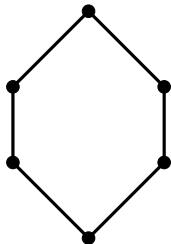
- $\mathbf{L} = (L, \leq)$.. (finite) lattice

Perspectivity

- $\mathbf{L} = (L, \leq)$.. (finite) lattice
- **edge**: (p, q) such that $p < q$ and no $p < r < q \rightsquigarrow \mathcal{E}(\mathbf{L})$
- **perspective**: $(p, q) \overline{\wedge} (r, s)$ such that $q \wedge r = p$ and $q \vee r = s$ (or $s \wedge p = r$ and $s \vee p = q$)

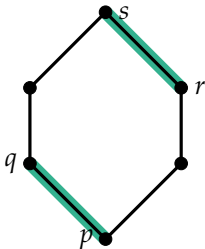
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Perspectivity

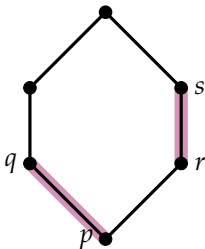
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perspective

Perspectivity

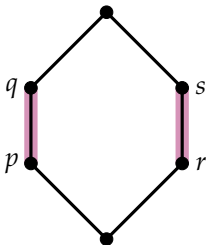
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not perspective

Perspectivity

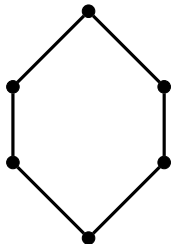
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not perspective

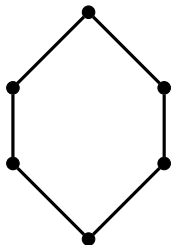
Irreducibility

- $\mathbf{L} = (L, \leq)$.. (finite) lattice



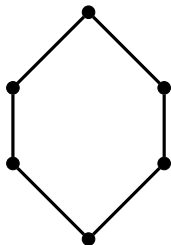
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Irreducibility

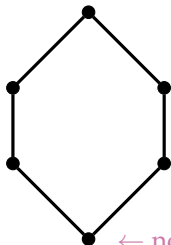
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← meet irreducible

Irreducibility

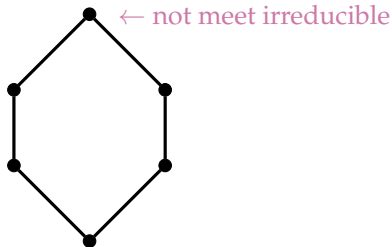
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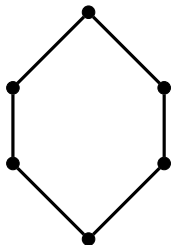
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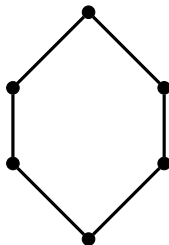
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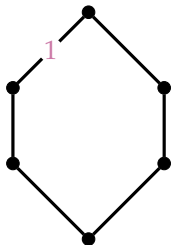
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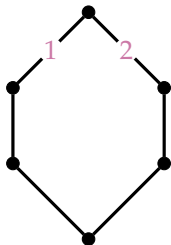
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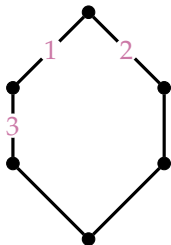
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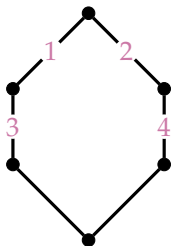
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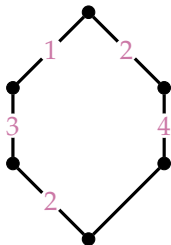
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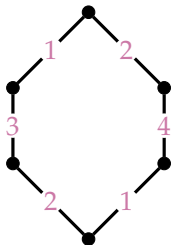
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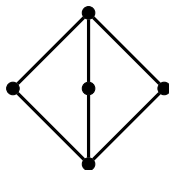
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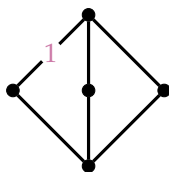
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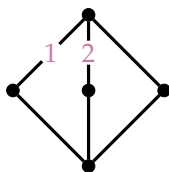
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- **M-determined**: for all $(p, q) \in \mathcal{E}(\mathbf{L})$ exists a unique $m \in \mathcal{M}(\mathbf{L})$ such that $(p, q) \overline{\wedge} (m, m^*)$



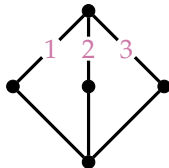
Irreducibility

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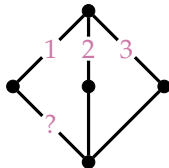
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Irreducibility

M-triangle

- $\mathbf{L} = (L, \leq)$.. (finite) lattice
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- **M-determined**: for all $(p, q) \in \mathcal{E}(\mathbf{L})$ exists a unique $m \in \mathcal{M}(\mathbf{L})$ such that $(p, q) \overline{\wedge} (m, m^*)$
- **perspectivity labeling**:

$$\lambda: \mathcal{E}(\mathbf{L}) \rightarrow \mathcal{M}(\mathbf{L}), \quad (p, q) \mapsto m$$

such that $(p, q) \overline{\wedge} (m, m^*)$

Meet-Semidistributive Lattices

- $\mathbf{L} = (L, \leq)$.. (finite) lattice

- **meet-semidistributive:**

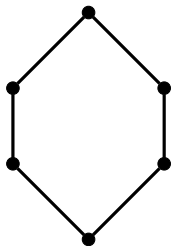
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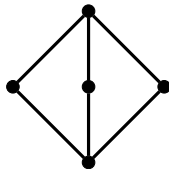
yes

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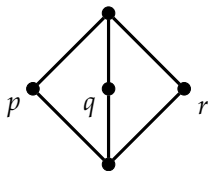
no

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$$p \wedge q = p \wedge r \quad \text{implies} \quad p \wedge q = p \wedge (q \vee r)$$

- if \mathbf{L} is meet-semidistributive, then

$$\lambda(p, q) \stackrel{\text{def}}{=} \max\{r \mid q \wedge r = p\}$$

is a perspectivity labeling

Proposition (✂, 2021)

Every meet-semidistributive lattice is M-determined.

Meet-Semidistributive Lattices

- $\mathbf{L} = (L, \leq)$.. (finite) lattice

- **meet-semidistributive:**

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$$\lambda(p, q) \stackrel{\text{def}}{=} \max\{r \mid q \wedge r = p\}$$

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Conjecture (, 2021)

A lattice is M-determined if and only if it is meet semidistributive.

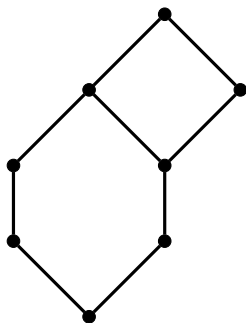
The M -Triangle

Posets from
Polytopes and
Refined Face
Enumeration

Henri Mühle

- $\mathbf{P} = (P, \leq)$.. poset

Questions

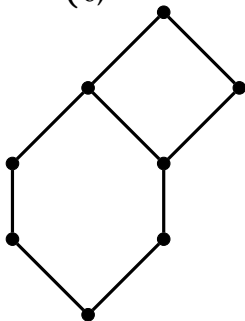


The M-Triangle

- $\mathbf{P} = (P, \leq)$.. poset

- **Möbius function:**

$$\mu_{\mathbf{P}}(p, q) \stackrel{\text{def}}{=} \begin{cases} 1, & p = q, \\ - \sum_{p \leq r < q} \mu_{\mathbf{P}}(p, r), & p < q, \\ 0, & \text{otherwise} \end{cases}$$

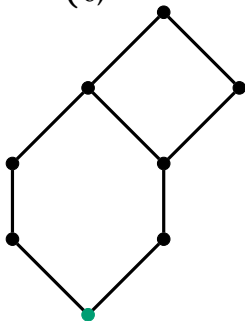


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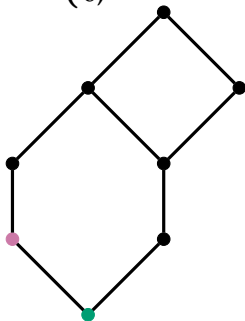
$$\mu(\bullet, \bullet) = 1$$

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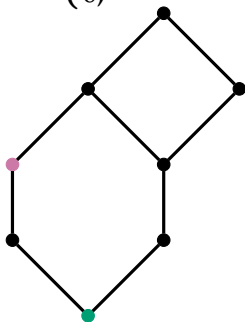
$$\mu(\bullet, \bullet) = -1$$

The M-Triangle

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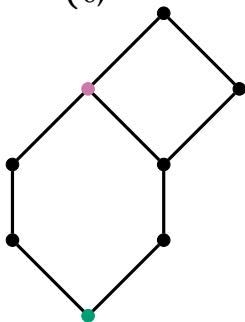
$$\mu(\bullet, \bullet) = 0$$

The M-Triangle

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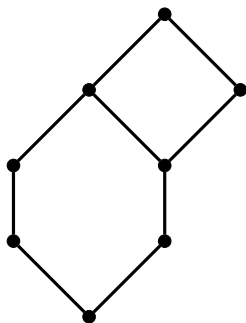
$$\mu(\bullet, \bullet) = 1$$

The M -Triangle

- $\mathbf{P} = (P, \leq)$.. ranked poset

Questions

- **M -triangle:** $M_{\mathbf{P}}(x, y) \stackrel{\text{def}}{=} \sum_{p, q \in P} \mu_{\mathbf{P}}(p, q) x^{\text{rk}(p)} y^{\text{rk}(q)}$

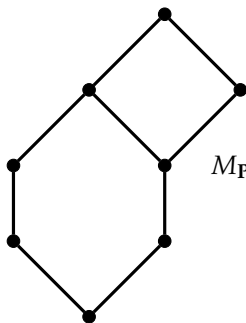


The M -Triangle

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Questions

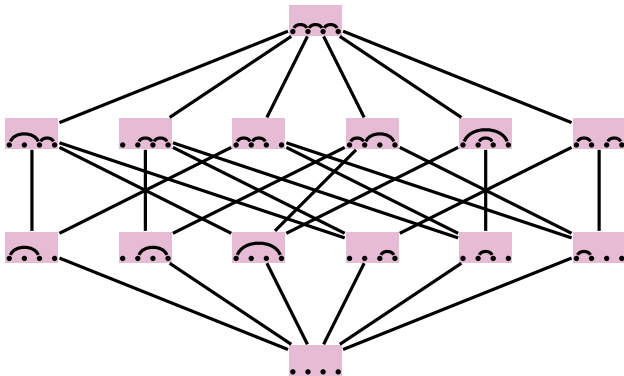
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$$\begin{aligned} M_{\mathbf{P}}(x, y) = & x^4 y^4 - 2x^3 y^4 + 2x^3 y^3 + x^2 y^4 \\ & - 3x^2 y^3 + 2x^2 y^2 - 2xy^2 \\ & + y^3 + 2xy - 2y + 1 \end{aligned}$$

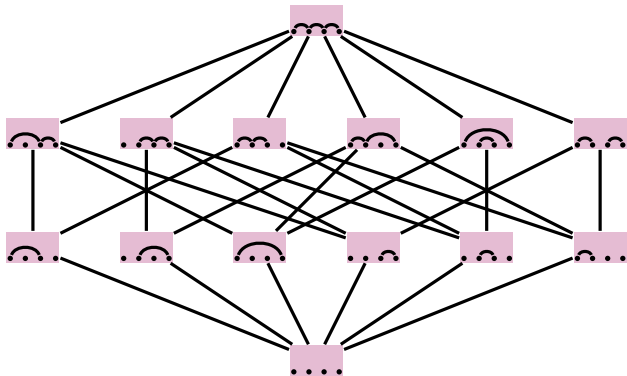
The M -Triangle

• $M_n(x, y) \stackrel{\text{def}}{=} M_{\text{Nonc}(n)}(x, y)$



The M -Triangle

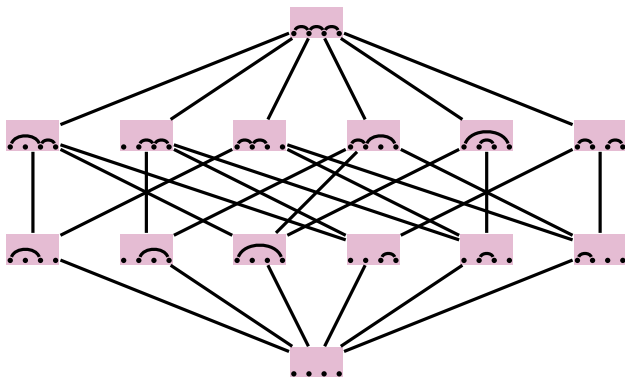
$$\bullet M_n(x, y) \stackrel{\text{def}}{=} M_{\text{Nonc}(n)}(x, y)$$



$$M_3(x, y) = x^3y^3 - 6x^2y^3 + 6x^2y^2 + 10xy^3 - 16xy^2 - 5y^3 + 6xy + 10y^2 - 6y + 1$$

The M-Triangle

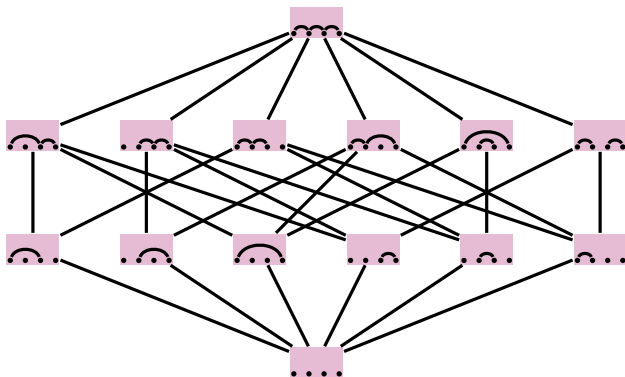
• $F_3(x, y) = 5x^3 + 5x^2y + 3xy^2 + y^3 + 10x^2 + 8xy + 3y^2 + 6x + 3y + 1$



$$M_3(x, y) = x^3y^3 - 6x^2y^3 + 6x^2y^2 + 10xy^3 - 16xy^2 - 5y^3 + 6xy + 10y^2 - 6y + 1$$

The M-Triangle

• $F_3(x, y) = 5x^3 + 5x^2y + 3xy^2 + y^3 + 10x^2 + 8xy + 3y^2 + 6x + 3y + 1$



$$M_3(x, y) = (xy - 1)^3 F_3 \left(\frac{1-y}{xy-1}, \frac{1}{xy-1} \right)$$

The M-Triangle

- $F_n(x, y) \stackrel{\text{def}}{=} \sum_{A \in \text{Clus}(n)} x^{\text{pos}(A)} y^{\text{neg}(A)}$
- $M_n(x, y) \stackrel{\text{def}}{=} \sum_{p, q \in \text{Nonc}(n)} \mu_{\text{Nonc}(n)}(p, q) x^{\text{rk}(p)} y^{\text{rk}(q)}$

Conjecture (F. Chapoton, 2004)

For $n \geq 1$,

$$M_n(x, y) = (xy - 1)^n F_n \left(\frac{1 - y}{xy - 1}, \frac{1}{xy - 1} \right).$$

The M -Triangle

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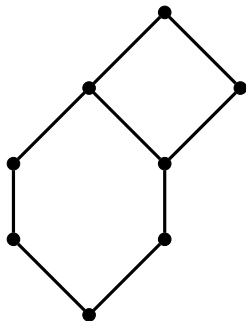
Theorem (C. Athanasiadis, 2007)

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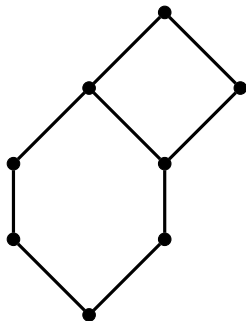
The Core Label Order

- $L = (L, \leq)$.. (finite) lattice



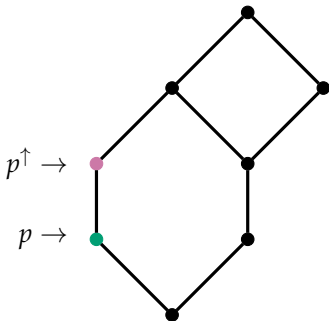
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- $\mathbf{L} = (L, \leq)$.. (finite) lattice, $p \in L$
- **nucleus**: $p^\uparrow \stackrel{\text{def}}{=} p \vee \bigvee \text{Succ}(p)$



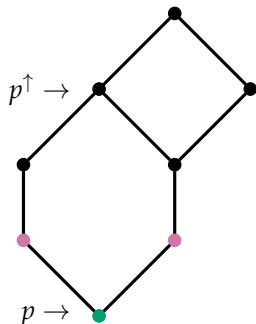
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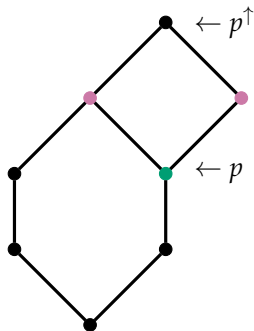
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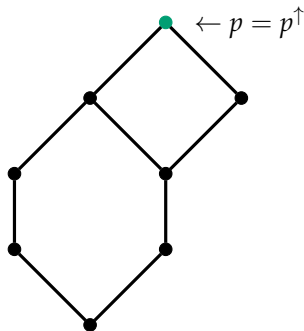
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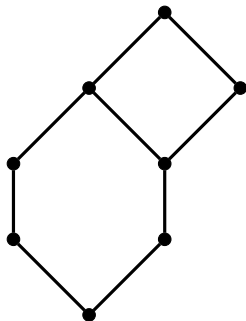
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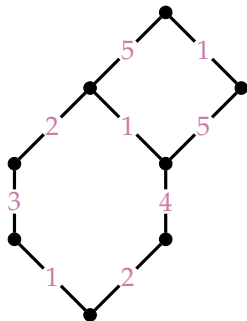
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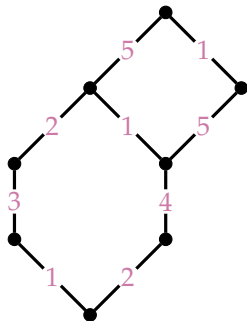
The Core Label Order

- $L = (L, \leq)$.. (finite) lattice, λ .. edge labeling



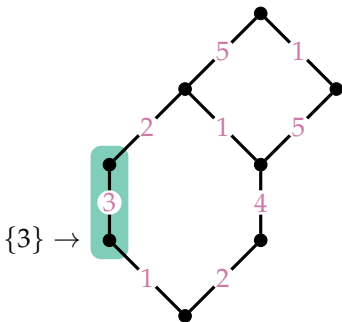
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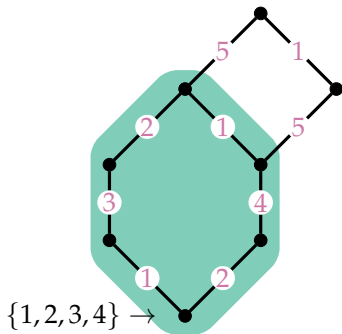
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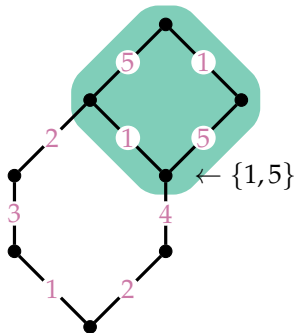
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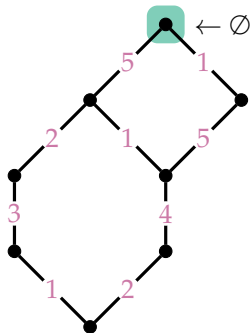
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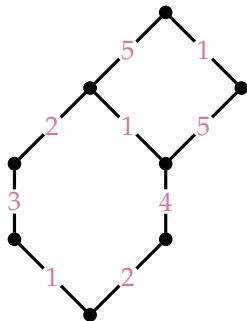
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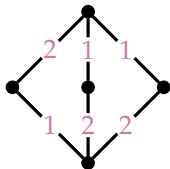
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The Core Label Order

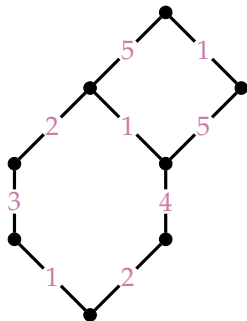
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not a core labeling

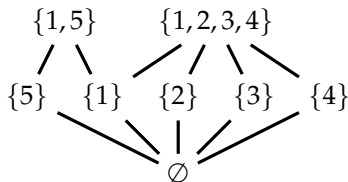
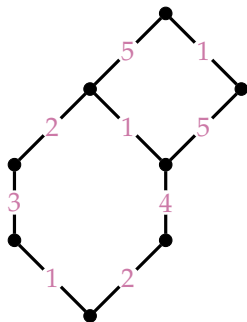
The Core Label Order

- $\mathbf{L} = (L, \leq)$.. (finite) lattice, λ .. edge labeling
- **core label order:** $\mathbf{CLO}_\lambda(\mathbf{L}) \stackrel{\text{def}}{=} (\mathbf{L}, \sqsubseteq)$,
where $p \sqsubseteq q$ if and only if $\Psi_\lambda(p) \subseteq \Psi_\lambda(q)$



The Core Label Order

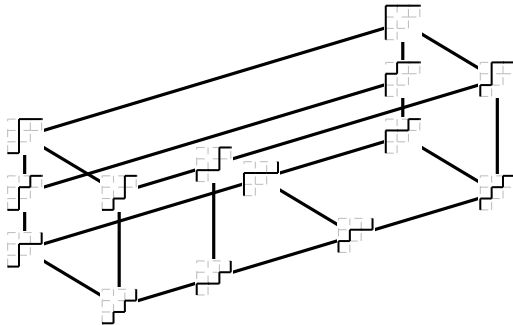
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The Core Label Order of the Tamari Lattice

Posets from
Polytopes and
Refined Face
Enumeration

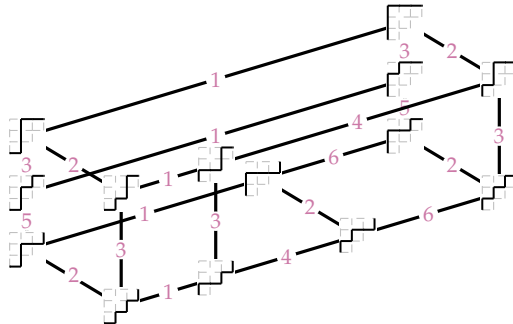
Henri Mühle



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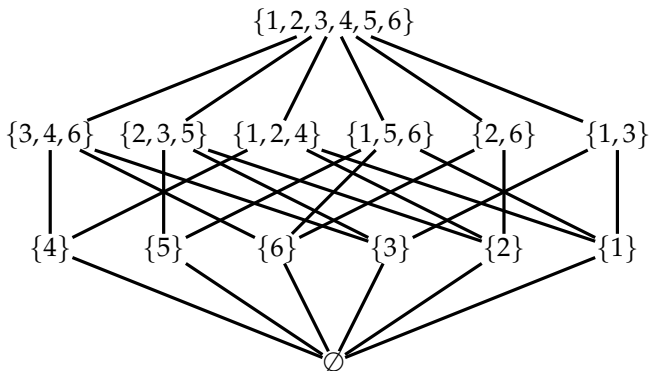
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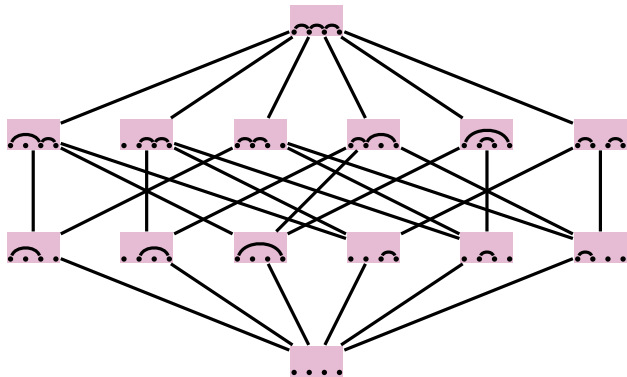
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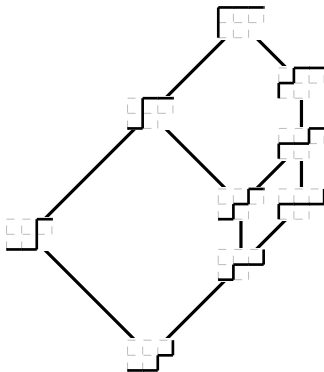
Theorem (N. Reading, 2011)

For $n > 0$, the core label order of $\mathbf{Tam}(n)$ is isomorphic to the noncrossing partition lattice $\mathbf{Nonc}(n)$.

The M -Triangle of ν -Tamari Lattices

- labeling edges of $\mathbf{Tam}(\nu)$ by **perspectivity**

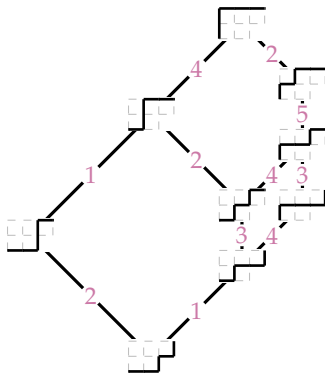
$\rightsquigarrow \mathbf{CLO}(\mathbf{Tam}(\nu))$



The M -Triangle of ν -Tamari Lattices

- labeling edges of $\mathbf{Tam}(\nu)$ by **perspectivity**

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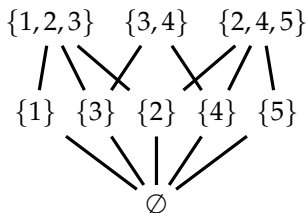
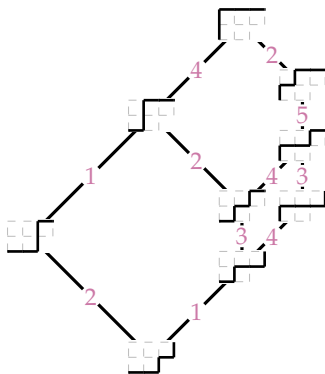


The M -Triangle of ν -Tamari Lattices

Perspectivity

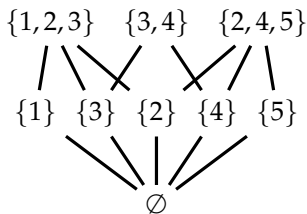
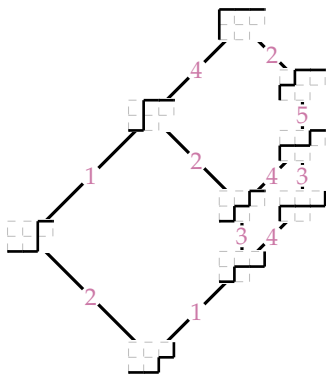
Irreducibility

- labeling edges of $\mathbf{Tam}(\nu)$ by **perspectivity**
 $\rightsquigarrow \mathbf{CLO}(\mathbf{Tam}(\nu))$



The M -Triangle of ν -Tamari Lattices

- labeling edges of $\mathbf{Tam}(\nu)$ by **perspectivity**
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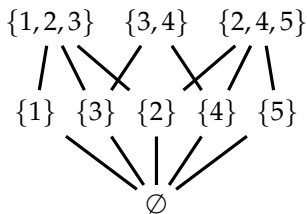
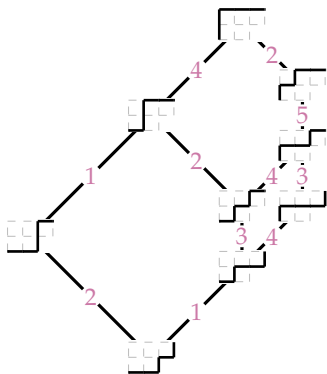
$$M_{EENEN}(x, y) = 3x^2y^2 - 8xy^2 + 5xy + 5y^2 - 5y + 1$$

The M -Triangle of ν -Tamari Lattices

Perspectivity

Irreducibility

- labeling edges of $\mathbf{Tam}(\nu)$ by **perspectivity**
 $\rightsquigarrow \mathbf{CLO}(\mathbf{Tam}(\nu))$
- $F_{EENEN}(x, y) = 5x^2 + 3xy + y^2 + 8x + 3y + 3$



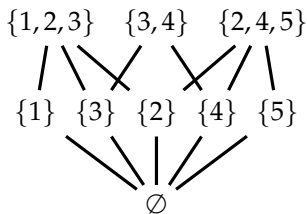
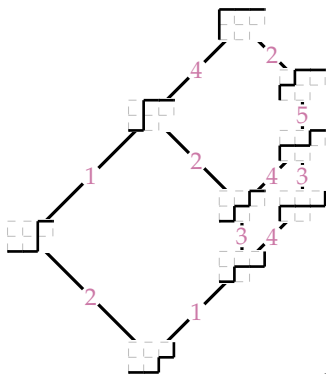
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The M -Triangle of ν -Tamari Lattices

Perspectivity

Irreducibility

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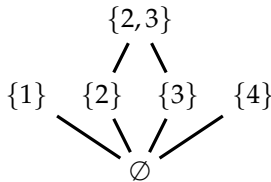
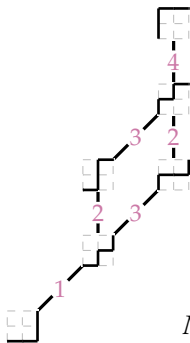
$$M_{EENEN}(x, y) = (xy - 1)^2 F_{EENEN} \left(\frac{1-y}{xy-1}, \frac{1}{xy-1} \right)$$

The M -Triangle of ν -Tamari Lattices

Perspectivity

Irreducibility

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 $\rightsquigarrow \mathbf{CLO}(\mathbf{Tam}(\nu))$



$$M_{EENN}(x, y) = x^2y^2 - 2xy^2 + 4xy + y^2 - 4y + 1$$

$$F_{EENN}(x, y) = 5x^2 + xy + 6x + 1$$

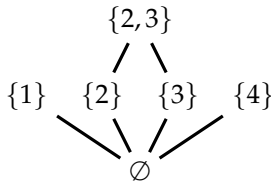
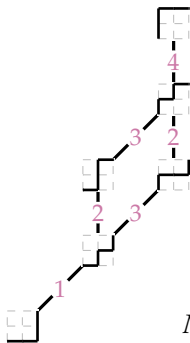
The M -Triangle of ν -Tamari Lattices

Perspectivity

Irreducibility

- labeling edges of $\mathbf{Tam}(\nu)$ by **perspectivity**

$\rightsquigarrow \mathbf{CLO}(\mathbf{Tam}(\nu))$



$$M_{EENN}(x, y) \neq (xy - 1)^2 F_{EENN} \left(\frac{1-y}{xy-1}, \frac{1}{xy-1} \right)$$

$$F_{EENN}(x, y) = 5x^2 + xy + 6x + 1$$

The M -Triangle of ν -Tamari Lattices

Posets from
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Enumeration

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Conjecture (✂, 2021)

Let ν be a northeast path. Then,

$M_\nu(x, y) = (xy - 1)^{\deg(\nu)} F_\nu \left(\frac{1-y}{xy-1}, \frac{1}{xy-1} \right)$ if and only if ν does not contain two consecutive east steps before two consecutive north steps.

The M -Triangle of ν -Tamari Lattices

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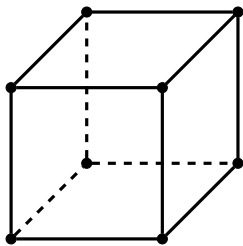
Conjecture (, 2021)

Let v be a northeast path. Then, $\text{Asso}(v)$ is pure if and only if v does not contain two consecutive east steps before two consecutive north steps.

Cubes

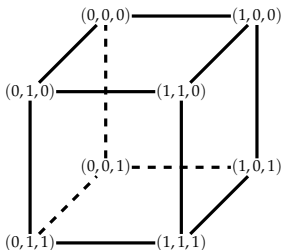
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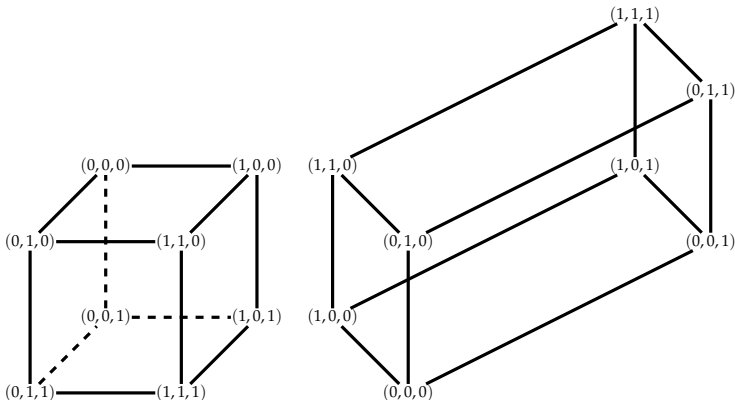
Cubes

- **binary tuple**: integer tuple (u_1, u_2, \dots, u_n) such that
 - $u_i \in \{0, 1\}$ $\rightsquigarrow \text{Bin}(n)$



Cubes

- **binary tuple**: integer tuple (u_1, u_2, \dots, u_n) such that
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- **Boolean lattice**:
 $\text{Bool}(n) \stackrel{\text{def}}{=} (\text{Bin}(n), \leq_{\text{comp}})$



Cubes

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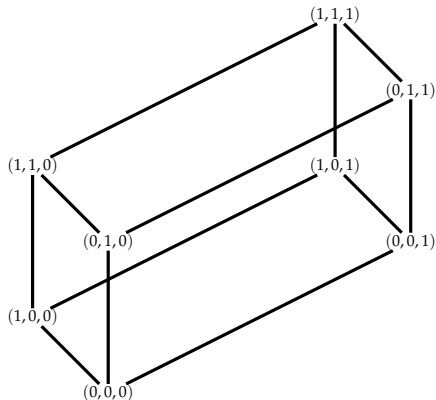
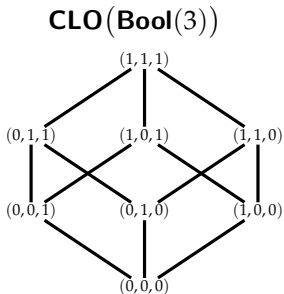
Theorem (Folklore)

For $n > 0$, $\mathbf{Bool}(n)$ is meet semidistributive.

Cubes

Theorem (✂, 2019)

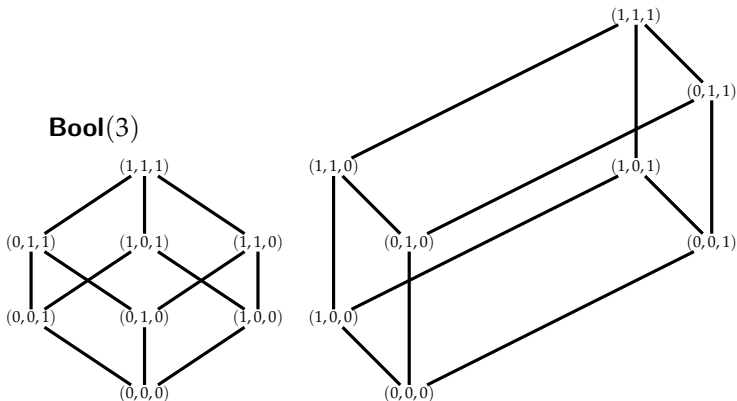
For $n > 0$, $\mathbf{CLO}(\mathbf{Bool}(n))$ is isomorphic to $\mathbf{Bool}(n)$.



Cubes

Theorem (✂, 2019)

For $n > 0$, $\mathbf{CLO}(\mathbf{Bool}(n))$ is isomorphic to $\mathbf{Bool}(n)$.



Cubes

- mark edges if perspective to **atoms**

Theorem (✂, 2020)

For $n > 0$, we have

$$F_{\mathbf{Bool}(n)}(x, y) = (x+y+1)^n,$$

$$H_{\mathbf{Bool}(n)}(x, y) = (xy+1)^n,$$

$$M_{\mathbf{Bool}(n)}(x, y) = (xy-y+1)^n.$$

Cubes

- mark all edges

Theorem (✂, 2020)

For $n > 0$, we have

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$$H_{\mathbf{Bool}(n)}(x, y) = (xy+1)^n,$$

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Stacked Cubes

- **chain poset:** $\mathbf{Chain}(n) \stackrel{\text{def}}{=} ([n], \leq)$
- let $\mathbf{L}(n_1, n_2, \dots, n_k) \stackrel{\text{def}}{=} \prod_{i=1}^k \mathbf{Chain}(n_i)$

Stacked Cubes

- **chain poset:** $\mathbf{Chain}(n) \stackrel{\text{def}}{=} ([n], \leq)$
- let $\mathbf{L}(n_1, n_2, \dots, n_k) \stackrel{\text{def}}{=} \prod_{i=1}^k \mathbf{Chain}(n_i)$

Theorem (✂, 2020)

For $n > 0$, we have

$$F_{\mathbf{L}(n_1, n_2, \dots, n_k)}(x, y) = \prod_{i=1}^k \left((n_i - 1)x + y + (n_i - 1) \right),$$

$$H_{\mathbf{L}(n_1, n_2, \dots, n_k)}(x, y) = \prod_{i=1}^k \left(xy + (n_i - 2)x + 1 \right),$$

$$M_{\mathbf{L}(n_1, n_2, \dots, n_k)}(x, y) = \prod_{i=1}^k \left((n_i - 1)xy - (n_i - 1)y + 1 \right).$$

Stacked Cubes

- **chain poset:** $\mathbf{Chain}(n) \stackrel{\text{def}}{=} ([n], \leq)$
- $\mathbf{L}(2, 2, \dots, 2) \cong \mathbf{Bool}(k)$

Corollary (✂, 2020)

For $n > 0$, we have

$$F_{\mathbf{L}(2,2,\dots,2)}(x, y) = \prod_{i=1}^k (x+y+1),$$

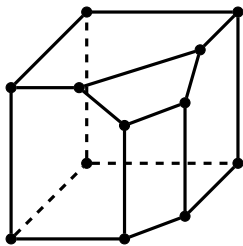
$$H_{\mathbf{L}(2,2,\dots,2)}(x, y) = \prod_{i=1}^k (xy+1),$$

$$M_{\mathbf{L}(2,2,\dots,2)}(x, y) = \prod_{i=1}^k (xy-y+1).$$

Freehedra

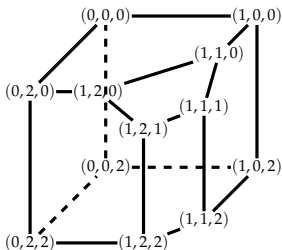
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Freehedra

- **triword**: integer tuple (u_1, u_2, \dots, u_n) such that $\rightsquigarrow \text{Tri}(n)$
 - $u_i \in \{0, 1, 2\}$
 - $u_1 \neq 2$
 - $u_i = 0$ implies $u_j \neq 1$ for all $j > i$



Freehedra

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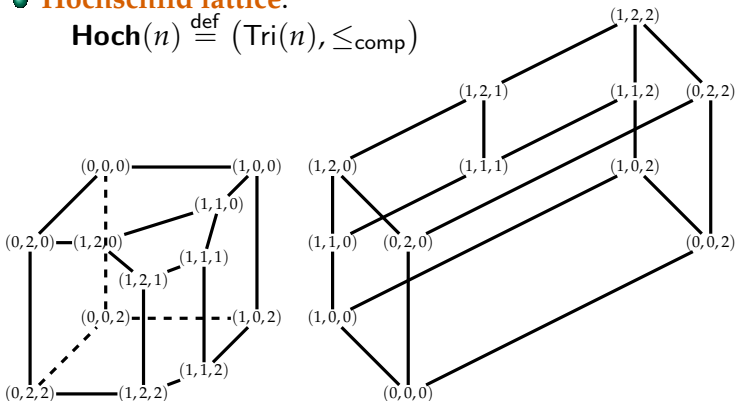
$$\rightsquigarrow \text{Tri}(n)$$

- $u_1 \neq 2$

- $u_i = 0$ implies $u_j \neq 1$ for all $j > i$

- **Hochschild lattice**:

$$\text{Hoch}(n) \stackrel{\text{def}}{=} (\text{Tri}(n), \leq_{\text{comp}})$$



Freehedra

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Theorem (C. Combe, 2020)

For $n > 0$, $\mathbf{Hoch}(n)$ is meet semidistributive.

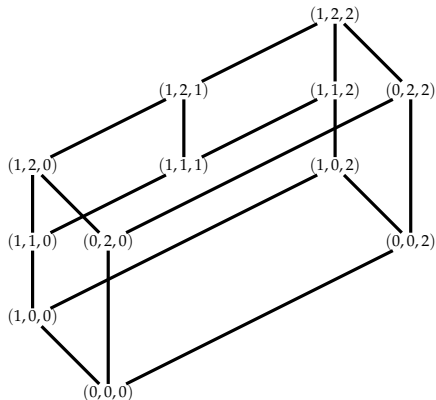
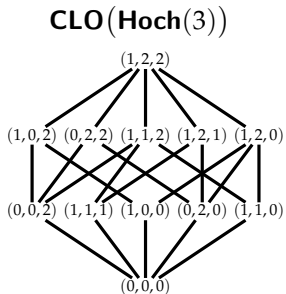
Freehedra

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Theorem (✂, 2020)

For $n > 0$, $\mathbf{CLO}(\mathbf{Hoch}(n))$ is isomorphic to $\mathbf{Shuf}(n-1, 1)$.



Freehedra

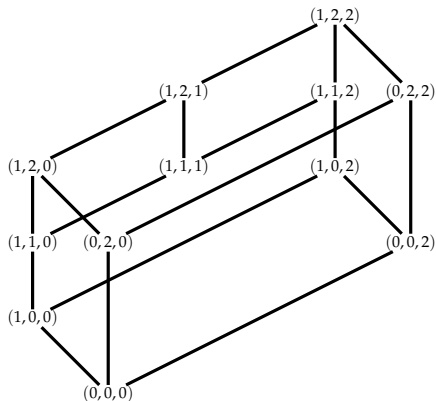
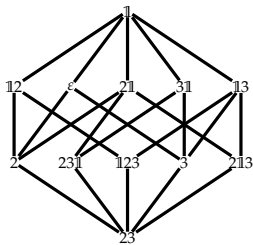
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Theorem (🌀, 2020)

For $n > 0$, $\mathbf{CLO}(\mathbf{Hoch}(n))$ is isomorphic to $\mathbf{Shuf}(n-1, 1)$.

Shuf(2, 1)



- mark edges if perspective to **atoms**

Theorem (✂, 2020)

For $n > 0$, we have

$$F_{\mathbf{Hoch}(n)}(x, y) = (x+y+1)^{n-2} (nx^2 + 2xy + (n+1)x + (y+1)^2),$$

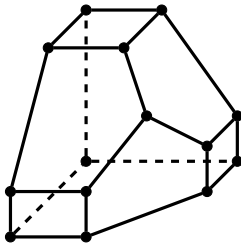
$$H_{\mathbf{Hoch}(n)}(x, y) = (xy+1)^{n-2} (x^2y^2 + 2xy + (n-1)x + 1),$$

$$M_{\mathbf{Hoch}(n)}(x, y) = (xy - y + 1)^{n-2} \\ \times \left((n+1)((x-1)y - xy^2) + (n+x^2)y^2 + 1 \right).$$

Associahedra

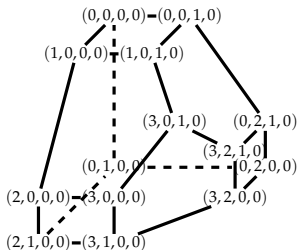
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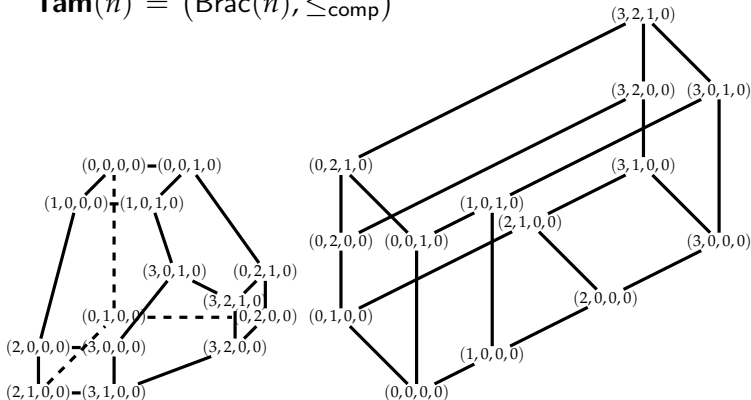
Associahedra

- **bracket vector:** integer tuple (u_1, u_2, \dots, u_n) such that
 - $u_i \in \{0, 1, \dots, n - i\}$ $\rightsquigarrow \text{Brac}(n)$
 - $u_{i+j} \leq u_i - j$ for all $j \in \{0, 1, \dots, u_i\}$



Associahedra

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- Tamari lattice:**
 $\text{Tam}(n) \stackrel{\text{def}}{=} (\text{Brac}(n), \leq_{\text{comp}})$



Associahedra

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- **Tamari lattice:**
 $\mathbf{Tam}(n) \stackrel{\text{def}}{=} (\text{Brac}(n), \leq_{\text{comp}})$

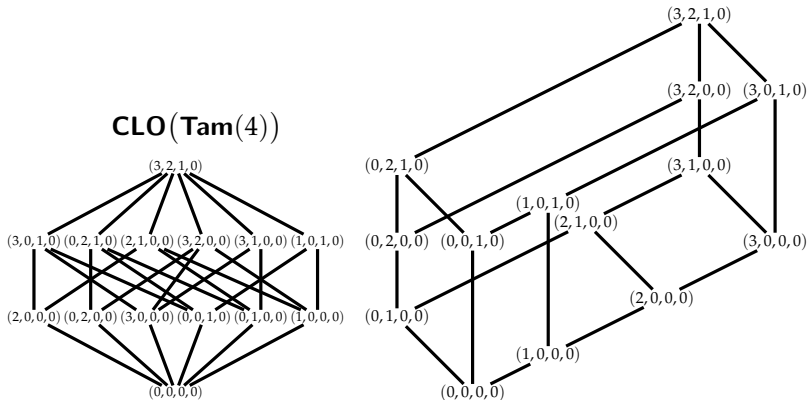
Theorem (A. Urquhart, 1978)

For $n > 0$, $\mathbf{Tam}(n)$ is meet semidistributive.

Associahedra

Theorem (N. Reading, 2011)

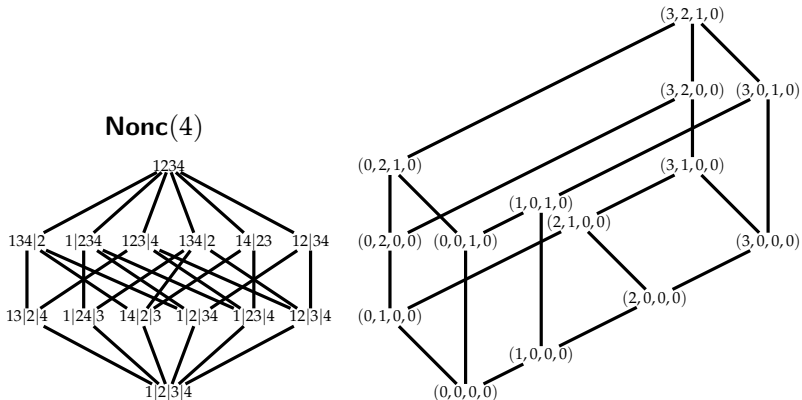
For $n > 0$, $\mathbf{CLO}(\mathbf{Tam}(n))$ is isomorphic to $\mathbf{Nonc}(n)$.



Associahedra

Theorem (N. Reading, 2011)

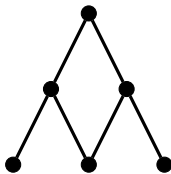
For $n > 0$, $\mathbf{CLO}(\mathbf{Tam}(n))$ is isomorphic to $\mathbf{Nonc}(n)$.



H -Triangles from Posets

- $\mathbf{P} = (P, \leq)$.. poset

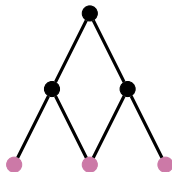
Questions



H -Triangles from Posets

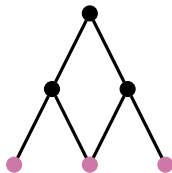
- $\mathbf{P} = (P, \leq)$.. poset; M .. minimal elements of \mathbf{P}

Questions



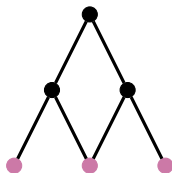
H-Triangles from Posets

- $\mathbf{P} = (P, \leq)$.. poset; M .. minimal elements of \mathbf{P} Questions
- **antichain**: pairwise incomparable elements $\rightsquigarrow \text{Anti}(\mathbf{P})$
- **\tilde{H} -triangle**: $\tilde{H}_{\mathbf{P}}(x, y) \stackrel{\text{def}}{=} \sum_{A \in \text{Anti}(\mathbf{P})} x^{|A|} y^{|A \cap M|}$



H-Triangles from Posets

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$$\tilde{H}_{\mathbf{P}}(x, y) = x^3 y^3 + 3x^2 y^2 + 2x^2 y + x^2 + 3xy + 3x + 1$$

H-Triangles from Posets

- $\mathbf{P} = (P, \leq)$.. poset; M .. minimal elements of \mathbf{P}

Questions

- **H-triangle:** $H_{\mathbf{P},\lambda}(x, y) \stackrel{\text{def}}{=} \sum_{p \in P} x^{\text{out}(p)} y^{\text{mrk}(p)}$

- **\tilde{H} -triangle:** $\tilde{H}_{\mathbf{P}}(x, y) \stackrel{\text{def}}{=} \sum_{A \in \text{Anti}(\mathbf{P})} x^{|A|} y^{|A \cap M|}$

Question (✂, 2021)

Let \mathbf{L} be a finite, meet-semidistributive lattice. Can we find an edge-labeling λ of \mathbf{L} and a poset \mathbf{P} such that $H_{\mathbf{L},\lambda}(x, y) = \tilde{H}_{\mathbf{P}}(x, y)$?

H-Triangles from Posets

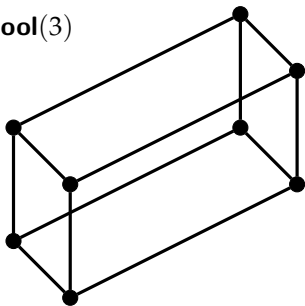
- $\mathbf{P} = (P, \leq)$.. poset; M .. minimal elements of \mathbf{P}

Questions

- **H-triangle:** $H_{\mathbf{P}, \lambda}(x, y) \stackrel{\text{def}}{=} \sum_{p \in P} x^{\text{out}(p)} y^{\text{mrk}(p)}$

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Bool(3)



H-Triangles from Posets

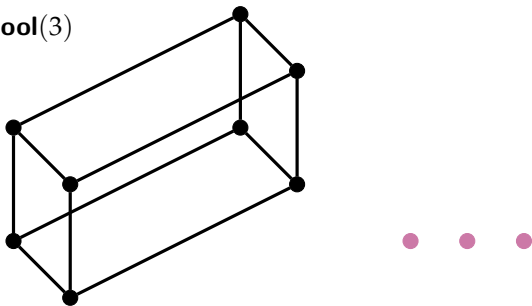
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Questions

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Bool(3)



H-Triangles from Posets

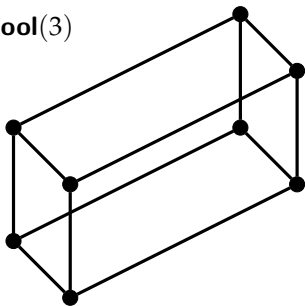
- $\mathbf{P} = (P, \leq)$.. poset; M .. minimal elements of \mathbf{P}

Questions

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Bool(3)



• • •

$$\tilde{H}(x, y) = (xy + 1)^3$$

H-Triangles from Posets

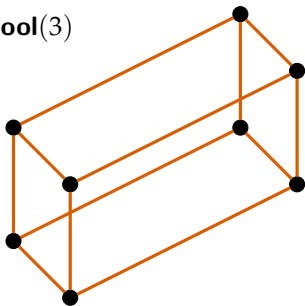
- $\mathbf{P} = (P, \leq)$.. poset; M .. minimal elements of \mathbf{P}

Questions

- **H-triangle:** $H_{\mathbf{P},\lambda}(x, y) \stackrel{\text{def}}{=} \sum_{p \in P} x^{\text{out}(p)} y^{\text{mrk}(p)}$

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Bool(3)



• • •

$$\tilde{H}(x, y) = (xy + 1)^3$$

H-Triangles from Posets

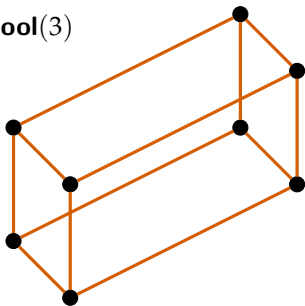
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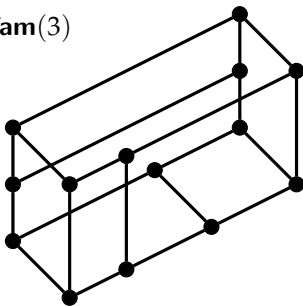
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Tam(3)



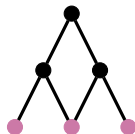
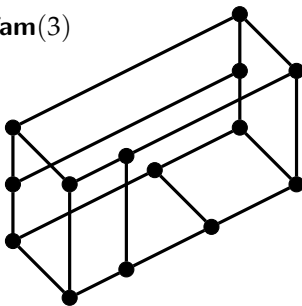
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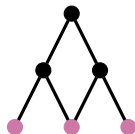
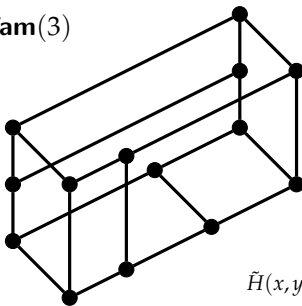
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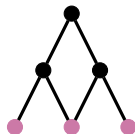
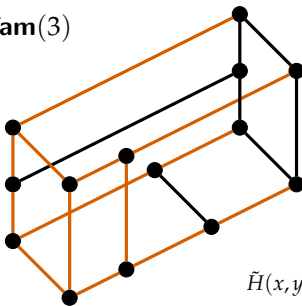
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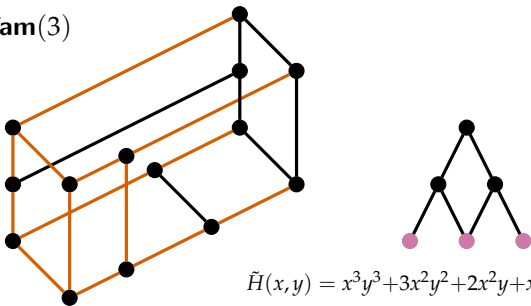
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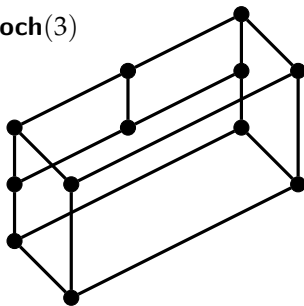
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Hoch(3)



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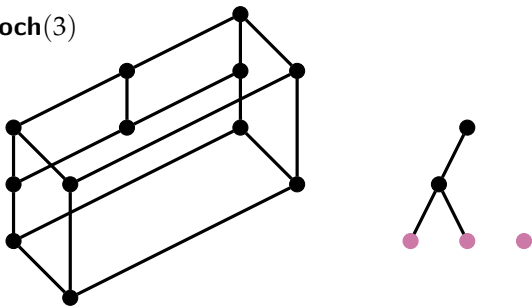
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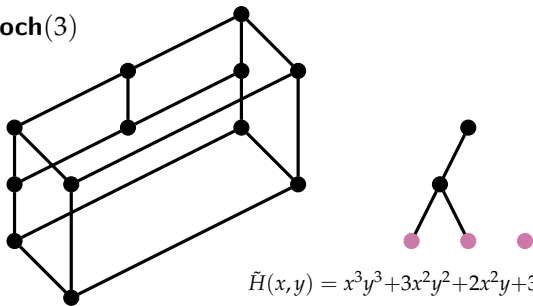
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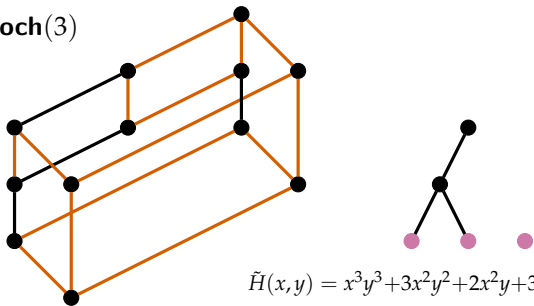
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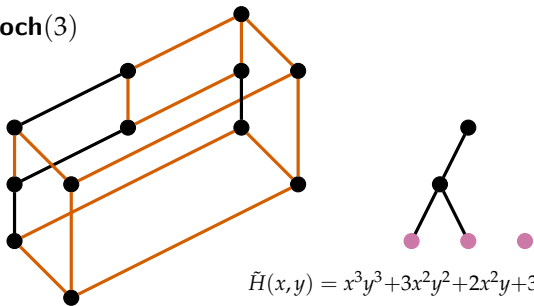
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