Proper Mergings of Stars and Chains
are Counted by Sums of Antidiagonals in Certain Convolution Arrays

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Outline

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Motivation

- let $(P, \leq_P)$ be a poset
- consider the elements of P as tasks
- for $p, p' \in P$, consider $p <_P p'$ as saying that the execution of p has to be finished before the execution of p' can begin
- thus, $(P, \leq_P)$ can be seen as a schedule, or an execution plan, and $\leq_P$ can be seen as a set of restrictions

- let $(Q, \leq_Q)$ be another poset
- How many different schedules exist such that

  $(P, \leq_P)$ and $(Q, \leq_Q)$ are executed "in parallel",
  no restrictions of $(P, \leq_P)$ and $(Q, \leq_Q)$ are violated or added
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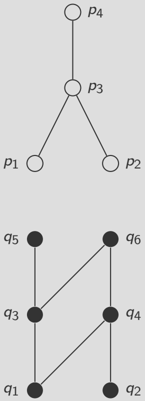
  - no two tasks are executed at the same time?
- we call such a schedule a proper merging of $(P, \leq_P)$ and $(Q, \leq_Q)$

Example



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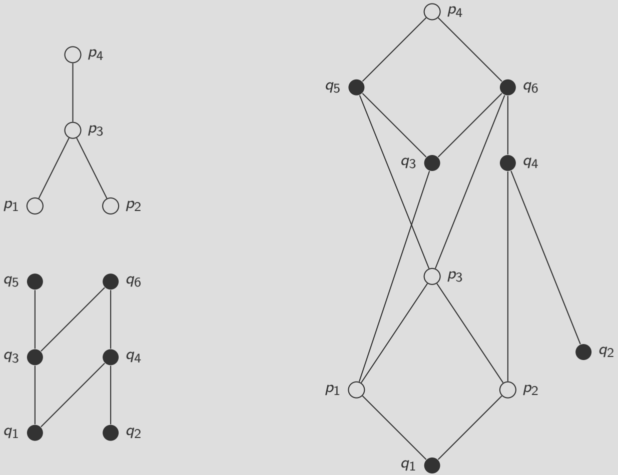

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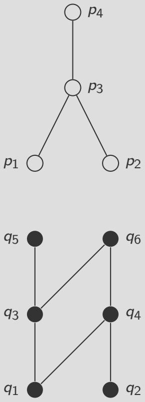

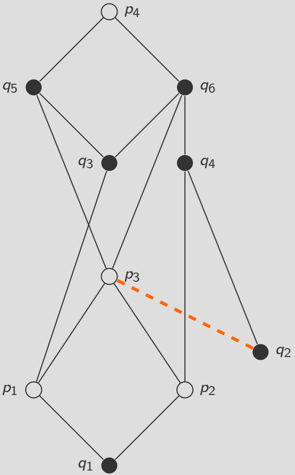

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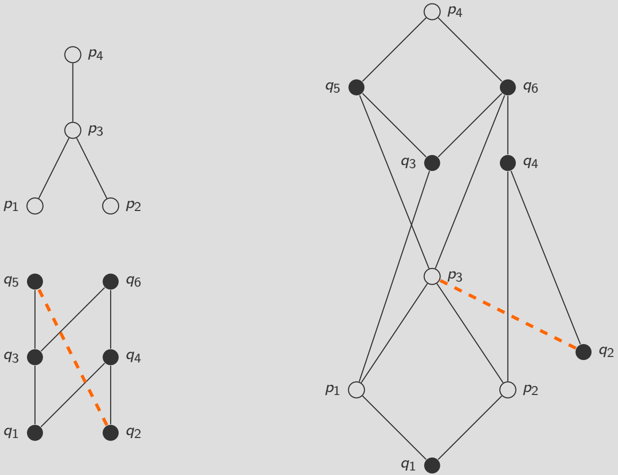

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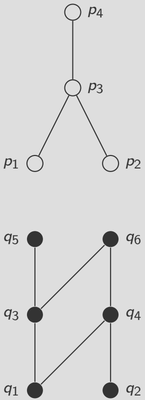

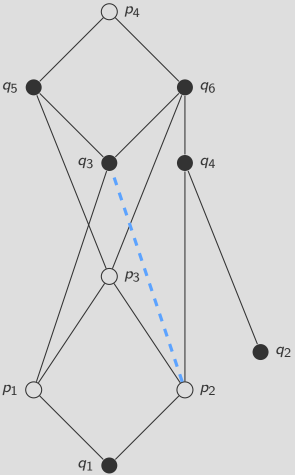

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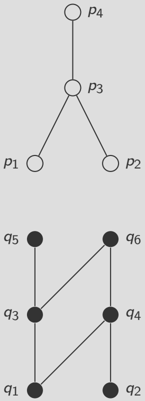

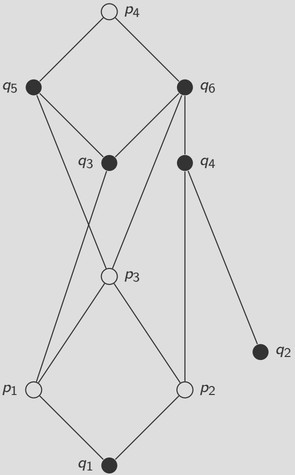

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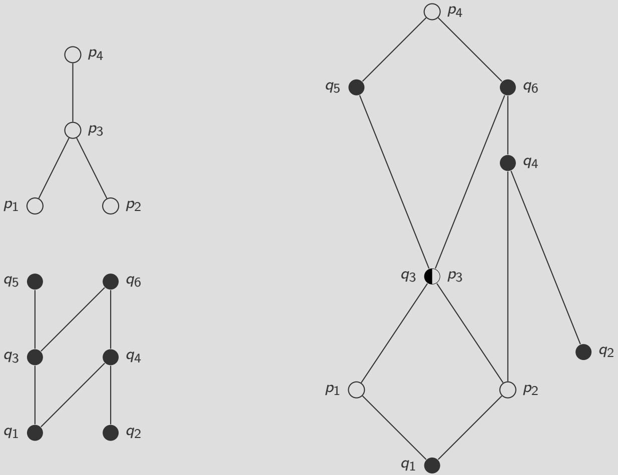


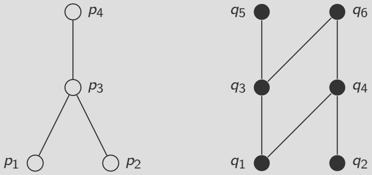
Outline

Characterization

- let $(G, M, J), (G', M', J')$ be formal contexts
- intent of (G, M, J): a set $A^J = \{m \in M \mid a\ J\ m \text{ for all } a \in A\}$ for $A \subseteq G$
- extent of (G, M, J): a set $B^J = \{g \in G \mid g\ J\ b \text{ for all } b \in B\}$ for $B \subseteq M$
- bond between (G, M, J) and (G', M', J'): a binary relation $R \subseteq G \times M'$ such that for all $g \in G$, the row g^R is an intent of (G', M', J'), and for all $m \in M'$, the column m^R is an extent of (G, M, J)

Example

	p_1	p_2	p_3	p_4	q_1	q_2	q_3	q_4	q_5	q_6
p_1	×		×	×						
p_2		×	×	×						
p_3			×	×						
p_4				×						
q_1					×		×	×	×	×
q_2						×		×		×
q_3							×		×	×
q_4								×		×
q_5									×	
q_6										×



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	p_1	p_2	p_3	p_4	q_1	q_2	q_3	q_4	q_5	q_6
p_1	×		×	×			×		×	×
p_2		×	×	×				×		×
p_3			×	×						×
p_4				×						
q_1					×		×	×	×	×
q_2						×		×		×
q_3							×		×	×
q_4								×		×
q_5									×	
q_6										×

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p_1	×		×	×			×		×	×
p_2		×	×	×				×		×
p_3			×	×			×			×
p_4				×						
q_1					×		×	×	×	×
q_2						×		×		×
q_3							×		×	×
q_4								×		×
q_5									×	
q_6										×

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p_2		×	×	×			×	×	×	×
p_3			×	×			×		×	×
p_4				×						
q_1					×		×	×	×	×
q_2						×		×		×
q_3							×		×	×
q_4								×		×
q_5									×	
q_6										×

Characterization

- let $(P,\leq_P)$ and $(Q,\leq_Q)$ be disjoint posets, and let $R\subseteq P\times Q$, and $T\subseteq Q\times P$
- for $p,q\in P\cup Q$, define $p\leftarrow_{R,T} q$ if and only if
$$p\leq_P q \quad\text{or}\quad p\leq_Q q \quad\text{or}\quad (p,q)\in R \quad\text{or}\quad (p,q)\in T$$
- merging of $(P,\leq_P)$ and $(Q,\leq_Q)$: a pair (R,T) such that $(P\cup Q,\leftarrow_{R,T})$ is a quasi-ordered set
- proper merging of $(P,\leq_P)$ and $(Q,\leq_Q)$: a merging (R,T) such that $R\cap T^{-1}=\emptyset$

Characterization

Proposition (Ganter, Meschke, M., 2011)

Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be disjoint posets, and let $R \subseteq P \times Q$ and $T \subseteq Q \times P$. The relation $\leftarrow_{R,T}$ is reflexive and transitive if and only if all of the following are satisfied:

1. *R is a bond between $(P, P, \not\geq_P)$ and $(Q, Q, \not\geq_Q)$,*
2. *T is a bond between $(Q, Q, \not\geq_Q)$ and $(P, P, \not\geq_P)$,*
3. *$R \circ T$ is contained in $\leq_P$,*
4. *$T \circ R$ is contained in $\leq_Q$.*

Moreover, $\leftarrow_{R,T}$ is antisymmetric if and only if $R \cap T^{-1} = \emptyset$.

- in other words, $(P \cup Q, \leftarrow_{R,T})$ is a poset if and only if (R, T) is a proper merging of $(P, \leq_P)$ and $(Q, \leq_Q)$

Characterization

Proposition (Ganter, Meschke, M., 2011)

Let $(P,\leq_P)$ *and* $(Q,\leq_Q)$ *be disjoint posets, and let* $R\subseteq P\times Q$ *and* $T\subseteq Q\times P$*. The relation* $\leftarrow_{R,T}$ *is reflexive and transitive if and only if all of the following are satisfied:*

1. R *is a bond between* $(P,P,\not\geq_P)$ *and* $(Q,Q,\not\geq_Q)$*,*
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- in other words, $(P\cup Q,\leftarrow_{R,T})$ is a poset if and only if (R,T) is a proper merging of $(P,\leq_P)$ and $(Q,\leq_Q)$

A Lattice Structure

- let $\mathfrak{M}_{P,Q}$ denote the set of mergings of $(P, \leq_P)$ and $(Q, \leq_Q)$
- define a partial order via

$$(R, T) \preceq (R', T') \quad \text{if and only if} \quad R \subseteq R' \text{ and } T \supseteq T',$$

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Theorem (Ganter, Meschke, M., 2011)

Let $(P,\leq_P)$ and $(Q,\leq_Q)$ be disjoint posets. The poset $(\mathfrak{M}_{P,Q},\preceq)$ is in fact a distributive lattice, where the least element is $(\emptyset,P\times Q)$ and the greatest element is $(P\times Q,\emptyset)$.
Moreover, $(\mathfrak{M}^{\bullet}_{P,Q},\preceq)$ is a distributive sublattice of the previous.

Outline

Enumeration

- Is it easy to determine the number of (proper) mergings of two posets $(P, \leq_P)$ and $(Q, \leq_Q)$?
- the number of (proper) mergings depends heavily on the structure of $(P, \leq_P)$ and $(Q, \leq_Q)$

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	1	18	142	723	2782	8796
	1	15	105	409	1764	5292

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		∘	∘ ∘	∘ ∘ ∘	∘ ∘ ∘ ∘	∘ ∘ ∘ ∘ ∘
	1	18	230	2676	30386	344748
	1	15	155	1443	12899	113235

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- the number of (proper) mergings depends heavily on the structure of $(P, \leq_P)$ and $(Q, \leq_Q)$
- we present the enumeration of two special cases:
  1. proper mergings of antichains and chains
  2. proper mergings of stars and chains

Outline

Preparation

- let $C = \{c_1, c_2, \ldots, c_n\}$ be a set and define $c_i \leq_{\mathfrak{c}} c_j$ if and only if $i \leq j$
- we notice that $c_i \not\geq_{\mathfrak{c}} c_j$ if and only if $i < j$, or equivalently $c_i <_{\mathfrak{c}} c_j$ for all $i, j \in \{1, 2, \ldots, n\}$
- thus, the extents of $(C, C, \not\geq_{\mathfrak{c}})$ are of the form $\{c_1, c_2, \ldots, c_k\}$ for some $k \in \{0, 1, \ldots, n\}$ and the intents are of the form $\{c_k, c_{k+1}, \ldots, c_n\}$ for some $k \in \{1, 2, \ldots, n+1\}$

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c_2		×	×	×
c_3			×	×
c_4				×

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c_2		×	×	×
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c_1		×	×	×
c_2			×	×
c_3				×
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- let $A = \{a_1, a_2, \ldots, a_n\}$ be a set and define $a_i =_{\mathfrak{a}} a_j$ if and only if $i = j$

- thus, the extents and intents of $(A, A, \neq_{\mathfrak{a}})$ are precisely the subsets of A

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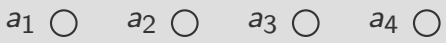

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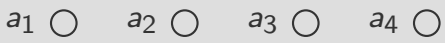

$=_{\mathfrak{a}}$	a_1	a_2	a_3	a_4
a_1	×			
a_2		×		
a_3			×	
a_4				×

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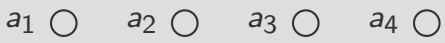

$=_{\mathfrak{a}}$	a_1	a_2	a_3	a_4
a_1	×			
a_2		×		
a_3			×	
a_4				×

$\neq_{\mathfrak{a}}$	a_1	a_2	a_3	a_4
a_1		×	×	×
a_2	×		×	×
a_3	×	×		×
a_4	×	×	×	

- thus, the extents and intents of $(A,A,\neq_{\mathfrak{a}})$ are precisely the subsets of A

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a_1	×			
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a_3			×	
a_4				×

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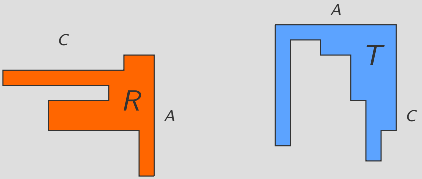
a_1		×	×	×
a_2	×		×	×
a_3	×	×		×
a_4	×	×	×	

- thus, the extents and intents of $(A, A, \neq_{\mathfrak{a}})$ are precisely the subsets of A

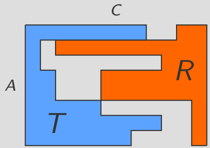
The Idea

- if (R, T) is a merging of $\mathfrak{a}$ and $\mathfrak{c}$, then R must be right-justified and T must be top-justified



The Idea

- if (R, T) is a proper merging of $\mathfrak{a}$ and $\mathfrak{c}$, then R and T must "fit together"



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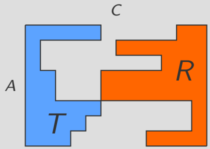

The Bijection

- complete bipartite digraph $\vec{K}_{m,n}$: a bipartite digraph with vertex set $V = V_1 \uplus V_2$, where $|V_1| = m$ and $|V_2| = n$, and edge set $\vec{E} = V_1 \times V_2$
- monotone coloring of a digraph: a map $\gamma : V \to \mathbb{N}$ with the property: if $(v_1, v_2) \in \vec{E}$, then $\gamma(v_1) \leq \gamma(v_2)$
- given a proper merging (R, T) of $\mathfrak{a}$ and $\mathfrak{c}$, define a monotone $(n+1)$-coloring γ of $\vec{K}_{m,m}$ as follows:

$$\gamma(v_i) = k \quad \text{if and only if} \quad \begin{cases} v_i \in V_1 & \text{and } a_i \; R \; c_j \\ & \text{for all } n+2-k \leq j \leq n \\ v_i \in V_2 & \text{and } c_j \; T \; a_i \\ & \text{for all } 1 \leq j \leq n+1-k \end{cases}$$

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- this is in fact a bijection!

The Enumeration

- the number of monotone n-colorings of $\vec{K}_{m_1,m_2}$ is known

Proposition (Jovović & Kilibarda, 2004)

Let $\kappa_n(\vec{K}_{m_1,m_2})$ denote the number of monotone n-colorings of $\vec{K}_{m_1,m_2}$. Then,

$$\kappa_n(\vec{K}_{m_1,m_2}) = \sum_{k=1}^{n}\Big((n+1-k)^{m_1} - (n-k)^{m_1}\Big)\cdot k^{m_2} = \sum_{k=1}^{n}\Big((n+1-k)^{m_2} - (n-k)^{m_2}\Big)\cdot k^{m_1}.$$

The Enumeration

- in view of the bijection from before, we obtain the following result

Theorem

The number $F_{\mathfrak{a}\mathfrak{c}}(m,n)$ of proper mergings of an m-antichain and an n-chain is given by

$$F_{\mathfrak{a}\mathfrak{c}}(m,n) = \kappa_{n+1}(\vec{K}_{m,m}) = \sum_{k=1}^{n+1}\Big((n+2-k)^m - (n+1-k)^m\Big)\cdot k^m.$$

- we need to evaluate the term "0^0" as zero, in order to cover the case $m=0$ correctly

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Outline

Preparation

- let $S=\{s_0,s_1,\ldots,s_n\}$ be a set and define $s_i\leq_{\mathfrak{s}} s_j$ if and only if $i=0$ or $i=j$
- thus, the extents and of $(S,S,\not\geq_{\mathfrak{s}})$ are either $\emptyset$ or $s_0\cup B$ for some $B\subseteq S\setminus\{s_0\}$, and the intents are either S or some subset of $S\setminus\{s_0\}$

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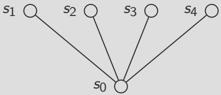

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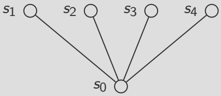

$\leq_{\mathfrak{s}}$	s_0	s_1	s_2	s_3	s_4
s_0	×	×	×	×	×
s_1		×			
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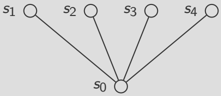

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s_0	×	×	×	×	×
s_1		×			
s_2			×		
s_3				×	
s_4					×

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s_0		×	×	×	×
s_1			×	×	×
s_2		×		×	×
s_3		×	×		×
s_4		×	×	×	

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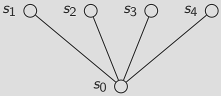

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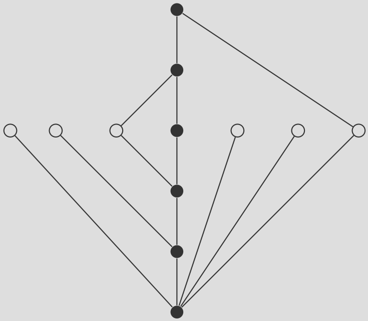
The Idea

- the poset $\big(S \setminus \{s_0\}, \leq_{\mathfrak{s}}\big)$ is an antichain
- we identify $A = S \setminus \{s_0\}$, and $\mathfrak{a} = (S \setminus \{s_0\}, \leq_{\mathfrak{s}})$
- if (R, T) is a proper merging of $\mathfrak{s}$ and $\mathfrak{c}$, then $(\bar{R}, \bar{T})$, with $\bar{R} = R \cap (A \times C)$ and $\bar{T} = T \cap (C \times A)$, is a proper merging of $\mathfrak{a}$ and $\mathfrak{c}$
- the map $\eta : \mathfrak{M}^{\bullet}_{\mathfrak{s},\mathfrak{c}} \to \mathfrak{M}^{\bullet}_{\mathfrak{a},\mathfrak{c}}, (R, T) \mapsto (\bar{R}, \bar{T})$ is a surjective lattice homomorphism
- thus, the lattice $\big(\mathfrak{M}^{\bullet}_{\mathfrak{a},\mathfrak{c}}, \preceq\big)$ is a quotient lattice of $\big(\mathfrak{M}^{\bullet}_{\mathfrak{s},\mathfrak{c}}, \preceq\big)$
- idea: count the fibers of η and determine the cardinality of each fiber

A Decomposition

- decompose the set $\mathfrak{M}^{\bullet}_{\mathfrak{a},\mathfrak{c}}$ with respect to three parameters:



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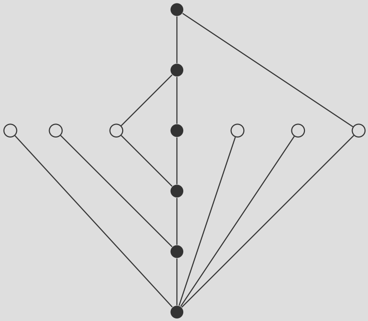

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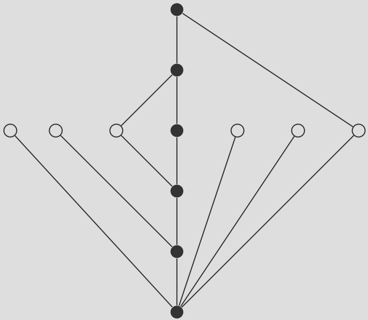

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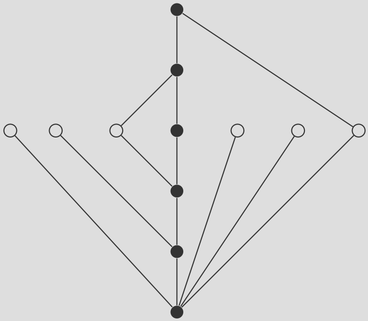

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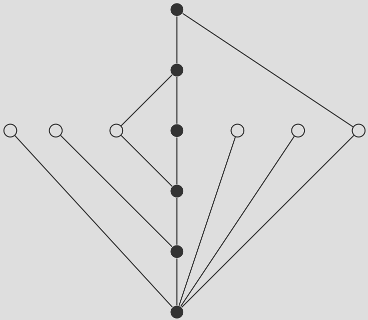

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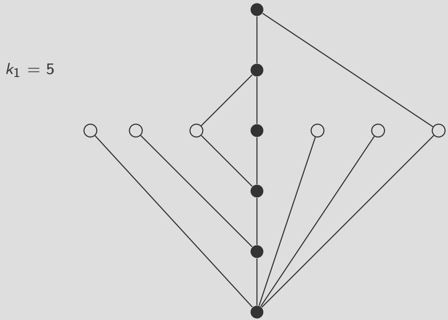

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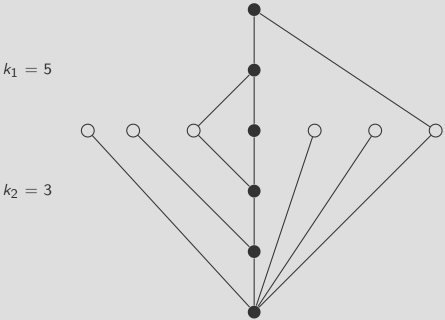

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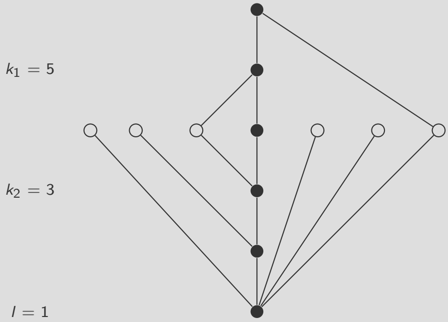

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- fix k_1, k_2, and l, and denote the set of all proper mergings of $\mathfrak{a}$ and $\mathfrak{c}$ which satisfy the previous constraints by $\mathfrak{M}^{\bullet}_{\mathfrak{a},\mathfrak{c}}(k_1, k_2, l)$
- let $V = V_1 \uplus V_2$ denote the vertex set of $\vec{K}_{m,m}$, let $F_{V_1}(m, n, k_1)$ resp. $F_{V_2}(m, k_2, l)$ denote the possibilities of coloring V_1 resp. V_2 (with respect to these constraints)
- we have

$$\left|\mathfrak{M}^{\bullet}_{\mathfrak{a},\mathfrak{c}}\right| = \sum \left|\mathfrak{M}^{\bullet}_{\mathfrak{a},\mathfrak{c}}(k_1, k_2, l)\right| = \sum F_{V_1}(m, n, k_1) \cdot F_{V_2}(m, k_2, l)$$

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Lemma

$F_{V_1}(m, n, k_1) = (n + 2 - k_1)^m - (n + 1 - k_1)^m.$

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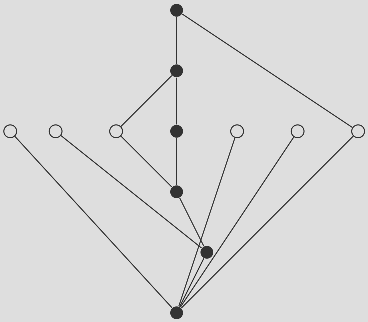
Lemma

$$F_{V_2}(m, k_2, l) = \begin{cases} 1, & \text{if } k_2 = l, \\ (k_2 - l + 1)^m - 2(k_2 - l)^m \\ \qquad + (k_2 - l - 1)^m, & \text{otherwise.} \end{cases}$$

The Fibers of η

- let $(R,T)\in\mathfrak{M}^{\bullet}_{\mathfrak{a},\mathfrak{c}}(k_1,k_2,l)$, and let (R_o,T_o) be the proper merging of $\mathfrak{s}$ and $\mathfrak{c}$ which is created from (R,T) by "adding s_0 to (R,T)"



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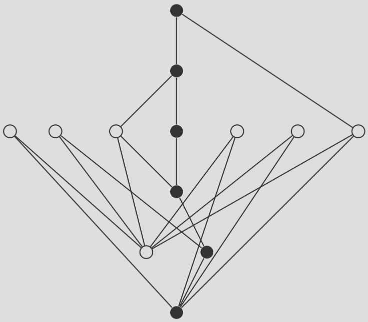

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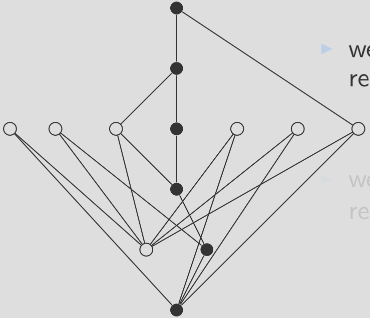

- we cannot add the following relations:
  - (c_j, s_0) for $k_1 \leq j \leq n$
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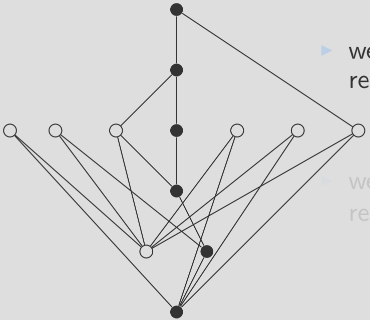

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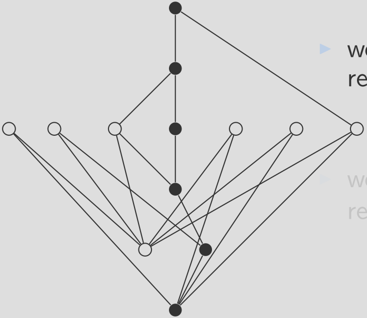

- we cannot add the following relations:
  - (c_j, s_0) for $k_1 \leq j \leq n$
  - (c_j, s_0) for $l < j < k_1$
- we can add all the other relations

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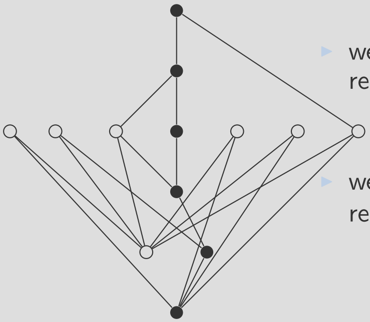

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The Fibers of η

- carefully counting the other possibilities yields the following result

Lemma

Let $(R, T) \in \mathfrak{M}^{\bullet}_{\mathfrak{a},\mathfrak{c}}(k_1, k_2, l)$. *Then,*

$$|\eta^{-1}(R, T)| = k_1(l + 1) - \binom{l+1}{2}.$$

The Enumeration

- in view of the previous reasoning, we obtain the following

$$\left|\mathfrak{M}^{\bullet}_{\mathfrak{s},\mathfrak{c}}\right| = \sum_{(R,T)\in\mathfrak{M}^{\bullet}_{\mathfrak{a},\mathfrak{c}}} \left|\eta^{-1}(R,T)\right|$$

$$= \sum_{k_1=1}^{n+1}\sum_{k_2=0}^{k_1-1}\sum_{l=0}^{k_2}\sum_{(R,T)\in\mathfrak{M}^{\bullet}_{\mathfrak{a},\mathfrak{c}}(k_1,k_2,l)} \left|\eta^{-1}(R,T)\right|$$

$$= \sum_{k_1=1}^{n+1}\sum_{k_2=0}^{k_1-1}\sum_{l=0}^{k_2} F_{V_1}(m,n,k_1)F_{V_2}(m,k_2,l)\left(k_1(l+1)-\binom{l+1}{2}\right)$$

$$= \sum_{k_1=1}^{n+1} F_{V_1}(m,n,k_1)\sum_{k_2=0}^{k_1-1}\sum_{l=0}^{k_2} F_{V_2}(m,k_2,l)\left(k_1(l+1)-\binom{l+1}{2}\right)$$

The Enumeration

- we finally obtain the result

Theorem

The number $F_{\mathfrak{x}}(m,n)$ of proper mergings of an m-star and an n-chain is given by

$$F_{\mathfrak{x}}(m,n)=\sum_{k=1}^{n+1}k^m(n-k+2)^{m+1}.$$

Another Interpretation of $F_{\mathfrak{x}}(m,n)$

- consider the maps $u_m(h) = h^m$ and $v_m(i,h) = (i-1+h)^m$
- define the convolution array $(a_{ij})_{i,j}$ of u_m and v_m via

$$\begin{aligned} a_{ij} &= \big(u_m(1), u_m(2), \ldots, u_m(j)\big) \star \big(v_m(i,1), v_m(i,2), \ldots, v_m(i,j)\big) \\ &= \sum_{k=1}^{j} u_m(k) \cdot v_m(i, j-k+1) \\ &= \sum_{k=1}^{j} \big(k(i+j-k)\big)^m \end{aligned}$$

Another Interpretation of $F_{\mathfrak{sc}}(m,n)$

- consider the maps $u_m(h) = h^m$ and $v_m(i,h) = (i-1+h)^m$
- the sum of the n-th antidiagonal of this array is

$$
\begin{aligned}
C(m,n) &= \sum_{l=1}^{n} a_{l,n-l+1} \\
&= \sum_{k=1}^{n} k^m (n-k+1)^{m+1}
\end{aligned}
$$

- we observe that $F_{\mathfrak{sc}}(m,n) = C(m,n+1)$

A Bijective Proof?

- let V_1, V_2, V_3 be sets with $|V_i| = m_i$ for $i \in \{1,2,3\}$
- consider the graph $\vec{K}_{m_1,m_2,m_3} = (V, \vec{E})$ with $V = V_1 \uplus V_2 \uplus V_3$ and $\vec{E} = (V_1 \times V_2) \cup (V_2 \times V_3)$
- let $\kappa_n(\vec{K}_{m_1,m_2,m_3})$ denote the number of monotone n-colorings of $\vec{K}_{m_1,m_2,m_3}$
- Christian Krattenthaler observed that $F_{\mathfrak{x}}(m,n) = \kappa_{n+1}(\vec{K}_{m,1,m+1})$

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Problem

Construct a bijection between the set of proper mergings of $\mathfrak{s}$ and $\mathfrak{c}$, and the set of monotone $(n+1)$-colorings of $\vec{K}_{m,1,m+1}$!

Outline

Outlook

- find enumeration formulas for the proper mergings of other families of posets
  - known: $|\mathfrak{M}^{\bullet}_{\mathfrak{c},\mathfrak{c}}|, |\mathfrak{M}^{\bullet}_{\mathfrak{a},\mathfrak{a}}|, |\mathfrak{M}^{\bullet}_{\mathfrak{a},\mathfrak{c}}|, |\mathfrak{M}^{\bullet}_{\mathfrak{s},\mathfrak{c}}|$
- investigate the relations between $\mathfrak{M}_{P,Q}$ and $\mathfrak{M}_{P',Q}$ under the assumption that P and P' are structurally related
  - we have seen that if P' is a subposet of P, then $(\mathfrak{M}_{P',Q},\preceq)$ is a quotient lattice of $(\mathfrak{M}_{P,Q},\preceq)$
  - for instance: if $P = P_1 \times P_2$, can $(\mathfrak{M}_{P,Q},\preceq)$ be explained via $(\mathfrak{M}_{P_1,Q},\preceq)$ and $(\mathfrak{M}_{P_2,Q},\preceq)$?

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Thank You.