# Parabolic Cataland

## Origins

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# Outline

# Catalan Families



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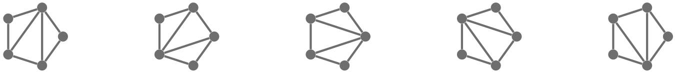
**triangulations** of an $(n+2)$-gon



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**Dyck paths** of length $2n$

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**231-avoiding permutations** of $[n]$

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**noncrossing partitions** of $[n]$

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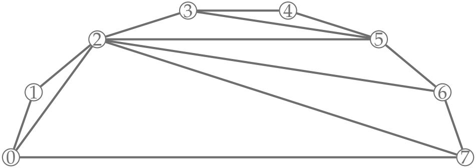

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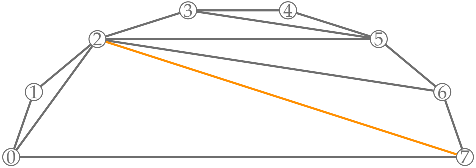

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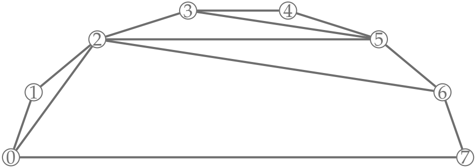

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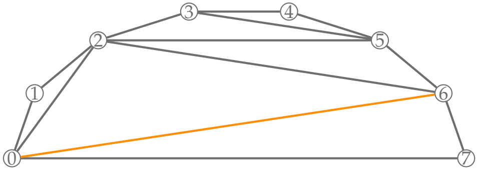

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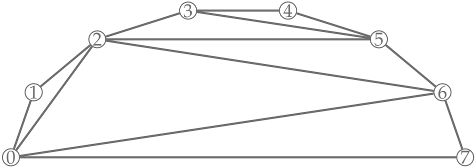

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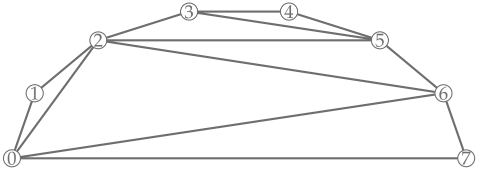

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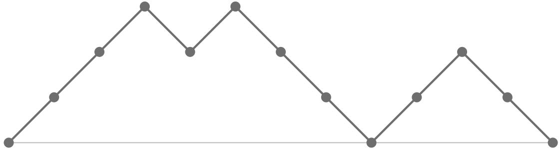

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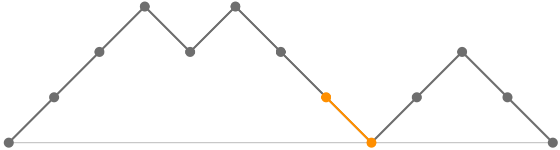

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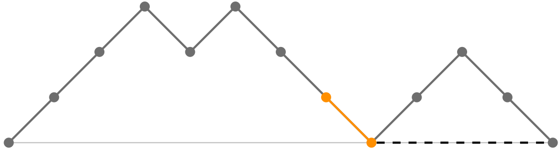

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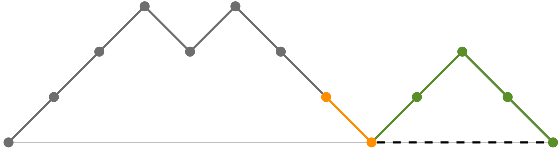

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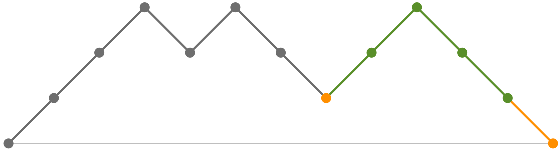

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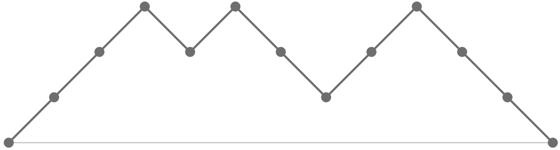

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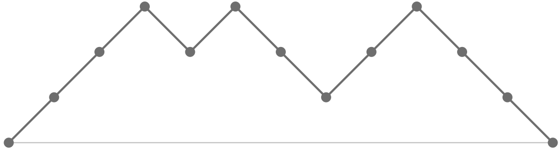

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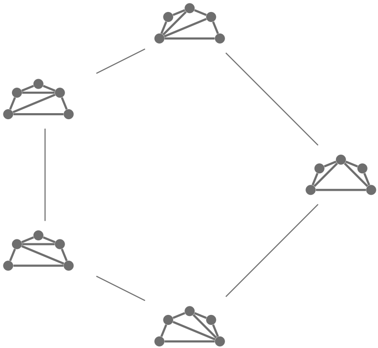

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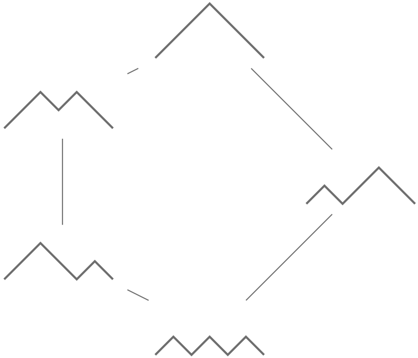

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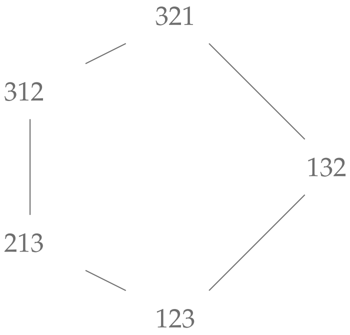

# Refinement Order on Noncrossing Partitions



- **noncrossing partition**: set partition of $[n]$ such that if $i < j < k < l$ and $i \sim k$ and $j \sim l$, then $i \sim j$ $\rightsquigarrow \mathrm{NC}_n$
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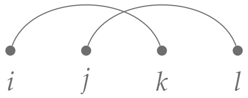

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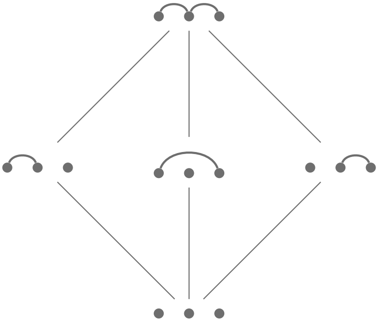

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**Theorem (N. Williams, 2013; N. Reading, 2015)**

*The set of cover inversions of a* 231*-avoiding permutation forms a noncrossing partition, and this correspondence is bijective.*

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- $X \subseteq \mathfrak{S}_n$ generating set closed under taking inverses
- ***X*-length**: word length with respect to alphabet $X \rightsquigarrow \ell_X$
- ***X*-postfix order**: $u \leq_X v$ if and only if
$$\ell_X(u) + \ell_X(vu^{-1}) = \ell_X(v)$$

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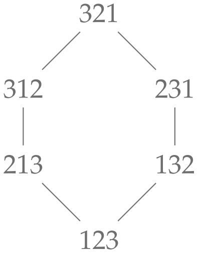

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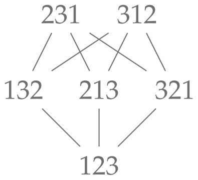

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**Theorem (A. Björner, 1980)**

*For $u, v \in \mathfrak{S}_n$ we have $u \leq_S v$ if and only if $u \leq_{\text{weak}} v$. Consequently, there is an isomorphism of lattices*

$$\bigl(\mathfrak{S}_n(231), \leq_S\bigr) \cong \mathcal{T}_n.$$

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**Theorem (P. Biane, 1997; T. Brady, 2001)**

*There is an isomorphism of lattices*

$$\Big(\big\{w \in \mathfrak{S}_n \mid w \leq_T (1\,2\,\ldots\,n)\big\}, \leq_T\Big) \cong \mathcal{NC}_n.$$

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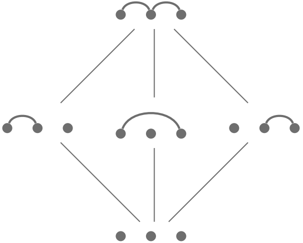

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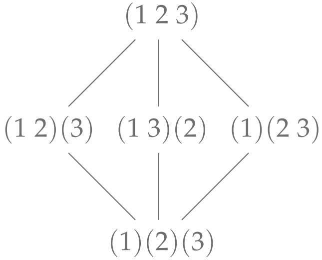

# The Root Poset



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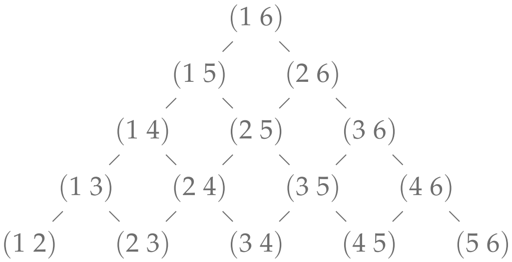
- **root poset**: $\Phi_n \stackrel{\text{def}}{=} (T_n, \preceq)$



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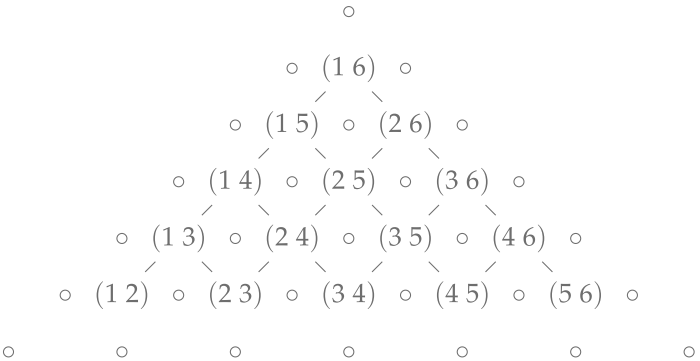

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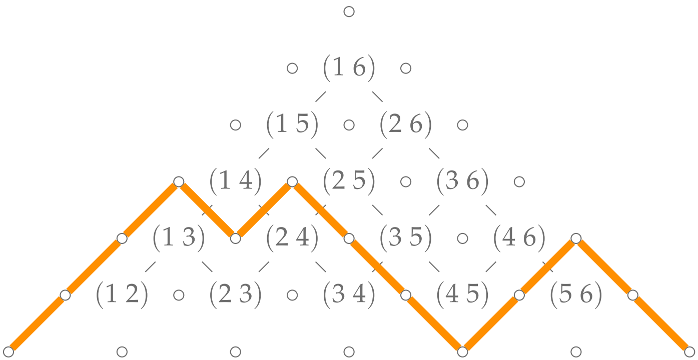

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**Theorem (A. Postnikov, 1997)**

*The set of order ideals in $\Phi_n$ is in bijection with $\mathcal{D}_n$.*

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# Two Observations



- for $(i,j),(k,l) \in T_n$ define

$$(i,j)+(k,l) \overset{\text{def}}{=} \begin{cases} (i,l), & \text{if } j=k,\\ \perp, & \text{otherwise} \end{cases}$$

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*If $w \in \mathfrak{S}_n(231)$ and $(i,j) + (j,k) \in \mathrm{Inv}(w)$, then $(i,j) \in \mathrm{Inv}(w)$.*

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*If $w \in \mathfrak{S}_n(231)$ and $(i,j)+(j,k) \in \mathrm{Inv}(w)$, then $(i,j) \in \mathrm{Inv}(w)$.*
*Inversion sets of 231-avoiding permutations are lexicographically "aligned".*

# Outline

# Coxeter Systems



- let $S$ be a finite set
- **Coxeter matrix**: $m\colon S\times S\to\{1,2,\ldots,\infty\}$ such that $m(s,t)=m(t,s)$ and $m(s,t)=1$ if and only if $s=t$
- **Coxeter system**: $(W,S)$ such that

$$W=\big\langle S\mid (st)^{m(s,t)}=\mathrm{id}\text{ if } m(s,t)\neq\infty\big\rangle_{\text{group}}$$

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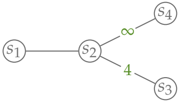
$$m=\begin{pmatrix}1&3&2&2\\3&1&4&\infty\\2&4&1&2\\2&\infty&2&1\end{pmatrix}\quad\longleftrightarrow$$



# Finite Coxeter Systems

$A_n, n \geq 1:$

$B_n, n \geq 2:$

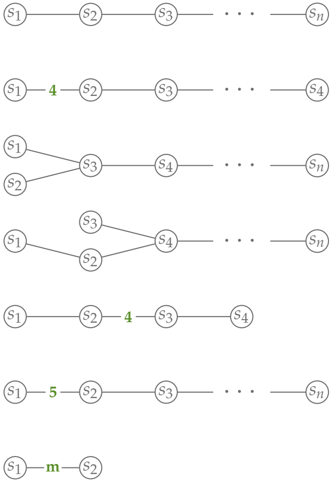
$D_n, n \geq 4:$

$E_n, 6 \leq n \leq 8:$

$F_4:$

$H_n, 2 \leq n \leq 4:$

$I_2(m), m \geq 6:$



# Reflection Groups



- $(\mathfrak{S}_n, S_n)$ is the Coxeter system of type $A_{n-1}$

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works only if $(W,S)$ is crystallographic

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- let $(W,S)$ be a finite Coxeter system with $|S|=n$
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- let $(W, S)$ be a finite Coxeter system

- $X \subseteq W$ generating set closed under taking inverses
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- canonical choices for $X$:
  - (left) **weak order**: $X = S$
  - **absolute order**: $X = T$

# Cataland

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**Definition (N. Reading, 2007)**

Let $c \in W$ be a Coxeter element. An element $w \in W$ is **$c$-aligned** if $t_\alpha \sqsubset_c t_{a\alpha+b\beta} \sqsubset_c t_\beta$ and $t_{a\alpha+b\beta} \in \mathrm{Inv}(w)$ imply $t_\alpha \in \mathrm{Inv}(w)$.

$\rightsquigarrow \mathrm{Align}(W,c)$

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*For every Coxeter element $c \in W$, we have*

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There is a long cycle $c \in \mathfrak{S}_n$ such that

$$\mathfrak{S}_n(231) = \mathrm{Align}(\mathfrak{S}_n, c).$$

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**Theorem (N. Reading, 2007)**

*For every Coxeter element $c \in W$, the poset $\bigl(\mathrm{Align}(W,c), \leq_S\bigr)$ is a lattice; the **c-Cambrian lattice**.*

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**Definition (T. Brady & C. Watt, 2002)**

Let $c \in W$ be a Coxeter element. An element $w \in W$ is **$c$-noncrossing** if $w \leq_T c$.

$\rightsquigarrow \mathrm{NC}(W, c)$

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**Theorem (D. Bessis, 2003)**

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- let $(W, S)$ be a finite Coxeter system, let $w \in W$
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- **cover inversion**: $t \in \mathrm{Inv}(w)$ such that $wt = sw$ for some $s \in S$ $\rightsquigarrow \mathrm{Cov}(w)$

**Theorem (N. Reading, 2007)**

*Let $w \in \mathrm{Align}(W, c)$ such that $\mathrm{Cov}(w) = \{t_1, t_2, \ldots, t_k\}$ with $t_1 \sqsubset_c t_2 \sqsubset_c \cdots \sqsubset_c t_k$. The product $t_1 t_2 \cdots t_k$ is c-noncrossing and this correspondence is bijective.*

# Outline

# Parabolic Cataland



- let $(W, S)$ be a finite Coxeter system
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$\rightsquigarrow$ there is a parabolic longest element $\rightsquigarrow w_\circ^J$
$\rightsquigarrow$ any Coxeter element $c$ induces a total order of $T^J$ via $\mathrm{Inv}\big(\mathbf{w}_\circ^J(c)\big)$ $\rightsquigarrow \sqsubset_c^J$

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**Definition (N. Williams, 2013)**

Let $c \in W$ be a Coxeter element. An element $w \in W^J$ is **$(W^J,c)$-aligned** if $t_\alpha \sqsubset_c^J t_{a\alpha+b\beta} \sqsubset_c^J t_\beta$ and $t_{a\alpha+b\beta} \in \mathrm{Cov}(w)$, then $t_\alpha \in \mathrm{Inv}(w)$.

$\rightsquigarrow \mathrm{Align}(W^J,c)$

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## Definition (N. Williams, 2013)

Let $c \in W$ be a Coxeter element. A product $t_1 t_2 \cdots t_k$ is **$(W^J, c)$-noncrossing** if there exists $w \in \mathrm{Align}(W^J, c)$ such that $\mathrm{Cov}(w) = \{t_1, t_2, \ldots, t_k\}$ with $t_1 \sqsubset_c^J t_2 \sqsubset_c^J \cdots \sqsubset_c^J t_k$.

$\rightsquigarrow \mathrm{NC}(W^J, c)$

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**Definition (N. Williams, 2013)**

The **parabolic root poset** is $(\Phi^+_{W^J}, \preceq)$, where $\Phi^+_{W^J}$ is the order filter of $(\Phi^+_W, \preceq)$ generated by the simple roots corresponding to the elements of $S \setminus J$.

# Parabolic Cataland



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**Definition (N. Williams, 2013)**

The **parabolic Catalan number**, denoted by $\mathrm{Cat}(W^J)$, is the number of order ideals in $(\Phi^+_{W^J}, \preceq)$.

# Parabolic Cataland



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**Conjecture (N. Williams, 2013)**

*Let $c \in W$ be a Coxeter element. We have*

$$\big|\mathrm{Align}(W^J, c)\big| = \mathrm{Cat}(W^J) = \big|\mathrm{NC}(W^J, c)\big|$$

*if and only if $(W, S)$ is of type $A_n$, $B_n$, $H_3$ or $I_2(m)$.*

# Parabolic Cataland



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**Conjecture (N. Williams, 2013)**

*Let $c\in W$ be a Coxeter element. The poset $\bigl(\mathrm{Align}(W^J,c),\leq_S\bigr)$ is a lattice.*

# Outline

# Parabolic Quotients of $\mathfrak{S}_n$



- consider $(\mathfrak{S}_n, S_n)$, and fix $c = (1\ 2\ \ldots\ n)$
- $w \in \mathfrak{S}_6$

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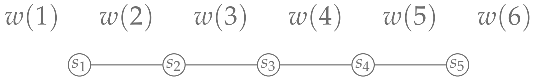

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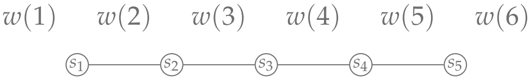

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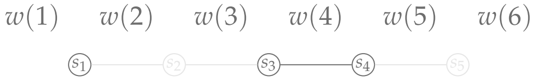

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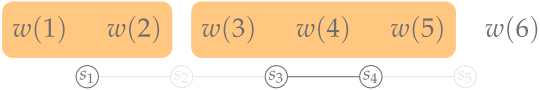

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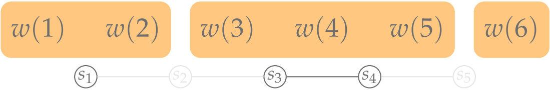

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$w(1) < w(2)$ $\quad$ $w(3) < w(4) < w(5)$ $\quad$ $w(6)$

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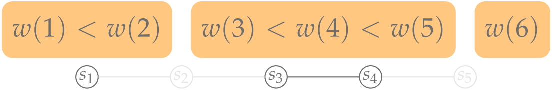

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- consider $(\mathfrak{S}_n, S_n)$, and fix $c = (1\ 2\ \ldots\ n)$
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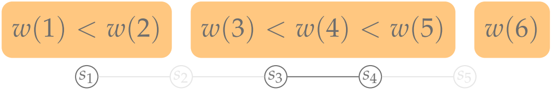

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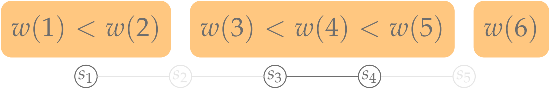
- $w \in \mathfrak{S}_6$
- $J = S_6 \setminus \{s_2, s_5\} \quad \longleftrightarrow \quad \alpha = (2, 3, 1)$
- subsets of $S_n$ correspond to compositions of $n$



# Parabolic Quotients of $\mathfrak{S}_n$



- consider $(\mathfrak{S}_n, S_n)$, and fix $c = (1\ 2\ \ldots\ n)$

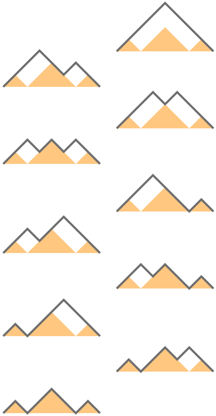
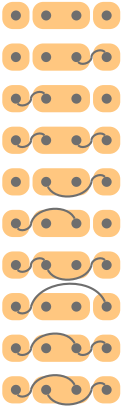
**Theorem ( & N. Williams, 2015)**

*For $c = (1\ 2\ \ldots\ n)$ and $J \subseteq S_n$ holds*

$$\bigl|\mathrm{Align}(\mathfrak{S}_n^J, c)\bigr| = \mathrm{Cat}(\mathfrak{S}_n^J) = \bigl|\mathrm{NC}(\mathfrak{S}_n^J, c)\bigr|.$$

# Parabolic Quotients of $\mathfrak{S}_n$

- let $n = 4, c = (1\,2\,3\,4), J = \{s_2\}$

| $\mathrm{Align}(\mathfrak{S}_4^J, c)$ | order ideals in $\left(\Phi^+_{\mathfrak{S}_4^J}, \preceq\right)$ | $\mathrm{NC}(\mathfrak{S}_4^J, c)$ |
|---|---|---|
| 1 2 3 4<br>1 2 4 3<br>2 1 3 4<br>2 1 4 3<br>1 3 4 2<br>3 1 2 4<br>3 2 4 1<br>4 1 2 3<br>4 1 3 2<br>4 2 3 1 |  |  |

# Parabolic Quotients of $\mathfrak{S}_n$



- consider $(\mathfrak{S}_n, S_n)$, and fix $c = (1\ 2\ \ldots\ n)$
- **parabolic Tamari lattice**: $(\mathrm{Align}(W^J, c), \leq_S)$ $\rightsquigarrow \mathcal{T}_n^J$

**Theorem (🌬 & N. Williams, 2015)**

*For $c = (1\ 2\ \ldots\ n)$ and $J \subseteq S_n$, the poset $\mathcal{T}_n^J$ is a lattice.*

# Parabolic Quotients of $\mathfrak{S}_n$



- consider $(\mathfrak{S}_n, S_n)$, and fix $c = (1\ 2\ \ldots\ n)$
- **parabolic Dyck paths**: order ideals in $\left(\Phi^+_{\mathfrak{S}_n^J}, \preceq\right) \quad \rightsquigarrow \mathcal{D}_n^J$

**Theorem (C. Ceballos, W. Fang & [windmill symbol], 2018)**

*For $c = (1\ 2\ \ldots\ n)$ and $J \subseteq S_n$, the lattices $\left(\mathrm{Align}(\mathfrak{S}_n^J), \leq_S\right)$ and $\left(\mathcal{D}_n^J, \leq_{\mathrm{rot}}\right)$ are isomorphic.*

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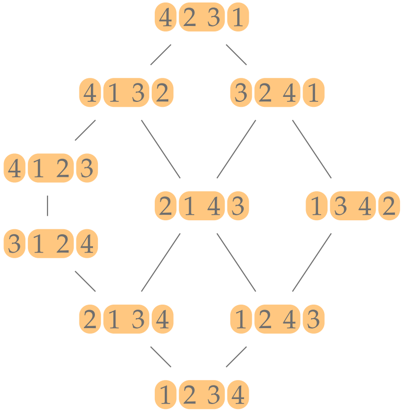

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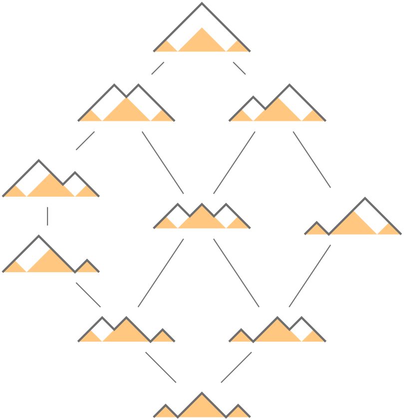

# Conclusion



- we have presented an algebraic framework to generalize classical Catalan families to parabolic quotients of finite Coxeter groups
- in type $A_n$ there are surprising connections to diagonal harmonics and Hopf algebras on pipe dreams

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- prospects:
  - combinatorial realizations for other Coxeter elements in type $A_n$ $\rightsquigarrow$ work in progress with V. Pilaud
  - combinatorial realizations for other types

# Thank You.