On the EL-Shellability of the Cambrian Lattices

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Motivation

- it is well-known that the Hasse diagram of the Tamari lattice corresponds to the 1-skeleton of the classical associahedron
- the Tamari lattice $\mathcal{T}_n$ can be realized as a lattice quotient of the weak order lattice of the Coxeter group A_n
- the bottom elements of each congruence class are precisely the 312-avoiding permutations
- Nathan Reading has generalized this construction to all finite Coxeter groups W and all Coxeter elements $\gamma \in W$
- he called the resulting lattices *Cambrian lattices*, denoted by C_γ
- this construction yields a generalized associahedron for all finite Coxeter groups

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- Björner and Wachs showed that $\mathcal{T}_n$ is EL-shellable and that each open interval of $\mathcal{T}_n$ is either contractible or spherical

  - it follows from a result by Nathan Reading that the open intervals of C_γ are either contractible or spherical

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  - Reading utilizes the fact that C_γ is the fan lattice of the Coxeter arrangement

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- however,
  - Thomas and Ingalls utilize the representation theory of Coxeter groups
  - Reading utilizes the fact that C_γ is the fan lattice of the Coxeter arrangement
- we give a direct, case-free proof of these properties, using the realization of C_γ in terms of γ-sortable elements

Outline

1. **Preliminaries**
   - Cambrian Lattices
   - EL-Shellability of Posets
2. **EL-Shellability of C_γ**
   - The Labeling
   - Main Result
3. **Applications**
   - Topology of C_γ
   - Subword Complexes

Outline

γ-Sorting Words

- let W be a finite Coxeter group of rank n, with simple generators $S = \{s_1, s_2, \ldots, s_n\}$
- consider the Coxeter element $\gamma = s_1 s_2 \cdots s_n$ and the half-infinite word $\gamma^\infty = s_1 s_2 \cdots s_n | s_1 s_2 \cdots s_n | s_1 \cdots$
- γ-sorting word of w: the reduced decomposition of $w \in W$ which is lexicographically first as a subword of γ^∞ among all reduced decompositions of w

γ-Sorting Words – Example

- let $W = A_4$ with $s_i = (i, i+1)$, and $\gamma = s_1 s_2 s_3 s_4$
- consider $w = s_1 s_4 s_3 s_4$
- there are eight reduced decompositions of w, namely

$$s_1 s_4 s_3 s_4, \quad s_4 s_1 s_3 s_4, \quad s_4 s_3 s_1 s_4, \quad s_4 s_3 s_4 s_1,$$
$$s_1 s_3 s_4 s_3, \quad s_3 s_1 s_4 s_3, \quad s_3 s_4 s_1 s_3, \quad s_3 s_4 s_3 s_1$$

- the decomposition $s_1 s_3 s_4 s_3$ is the lexicographically first subword of γ^∞ among these words, and thus the γ-sorting word of w

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$$s_1s_2s_3s_4 \big| s_1s_2s_3s_4 \big| s_1s_2s_3s_4$$

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$$\mathbf{s_1} s_2 s_3 \mathbf{s_4} \big| s_1 s_2 \mathbf{s_3 s_4} \big| s_1 s_2 s_3 s_4$$

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γ-Sortable Words

- write the γ-sorting word of w as follows

$$w = s_1^{\delta_{1,1}} s_2^{\delta_{1,2}} \cdots s_n^{\delta_{1,n}} | s_1^{\delta_{2,1}} s_2^{\delta_{2,2}} \cdots s_n^{\delta_{2,n}} | \cdots | s_1^{\delta_{l,1}} s_2^{\delta_{l,2}} \cdots s_n^{\delta_{l,n}},$$

  where $\delta_{i,j} \in \{0,1\}$ for $1 \leq i \leq l$ and $1 \leq j \leq n$
- i-th block of w: the set $b_i = \{s_j \mid \delta_{i,j} = 1\} \subseteq S$, where $i \in \{1,2,\ldots,l\}$
- γ-sortable word: a word $w \in W$ satisfying $b_1 \supseteq b_2 \supseteq \cdots \supseteq b_l$

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- γ-sortable word: a word $w \in W$ satisfying $b_1 \supseteq b_2 \supseteq \cdots \supseteq b_l$

- the γ-sorting word $w = s_1 s_3 s_4 | s_3$ has $b_1 = \{s_1, s_3, s_4\}$ and $b_2 = \{s_3\}$ and is thus γ-sortable
- the γ-sorting word $v = s_1 s_3 s_4 | s_2$ is not

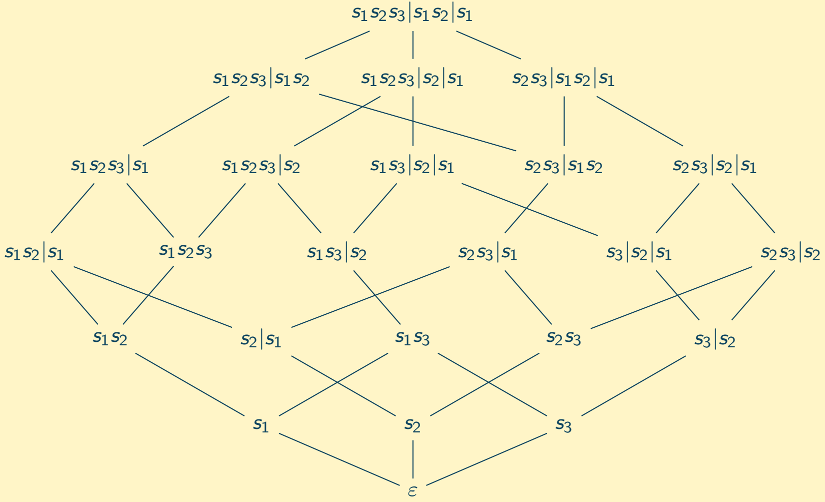
Cambrian Lattices

Theorem (Reading, 2005)

Let γ be a Coxeter element of a finite Coxeter group W. The γ-sortable elements of W constitute a sublattice of the weak order on W.

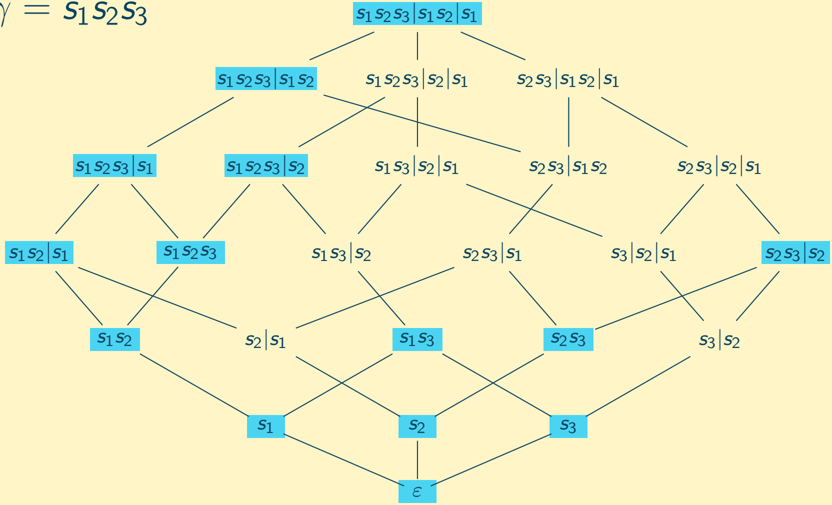
- consider the map $\pi_\gamma : W \to W, w \mapsto \pi_\gamma(w)$ that maps w to the largest γ-sortable element below it
- the fibers of π_γ induce a lattice congruence θ_γ on the weak order on W
- Cambrian lattice C_γ: the lattice quotient W/θ_γ

Cambrian Lattices – Example



Cambrian Lattices – Example

$\gamma = s_1 s_2 s_3$



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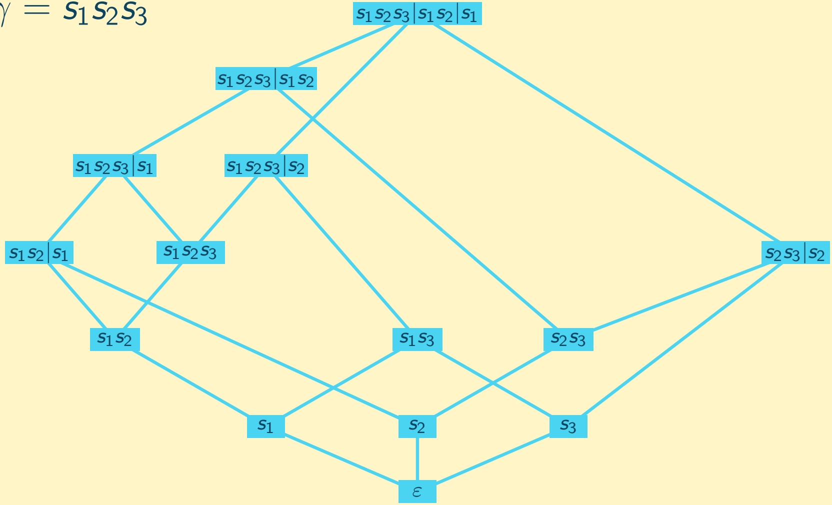

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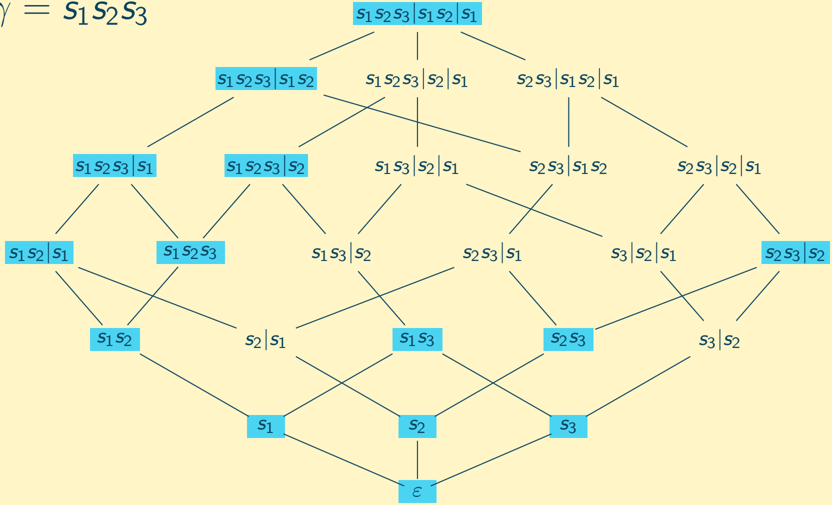

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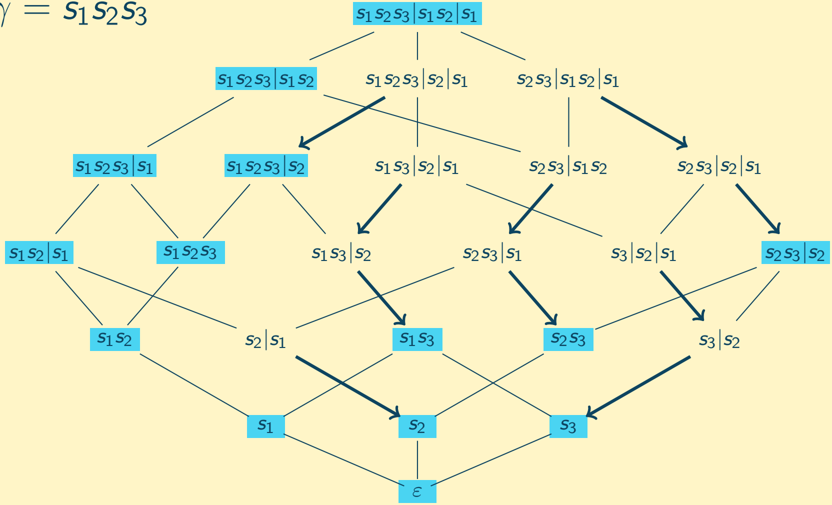

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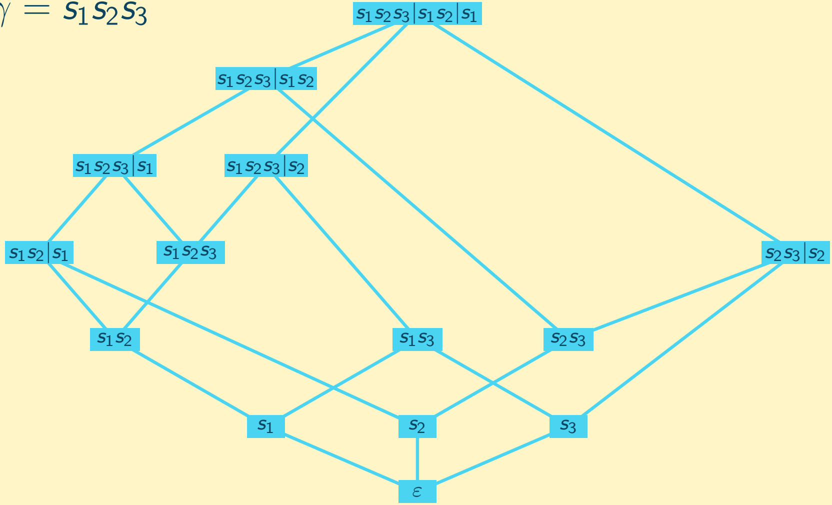

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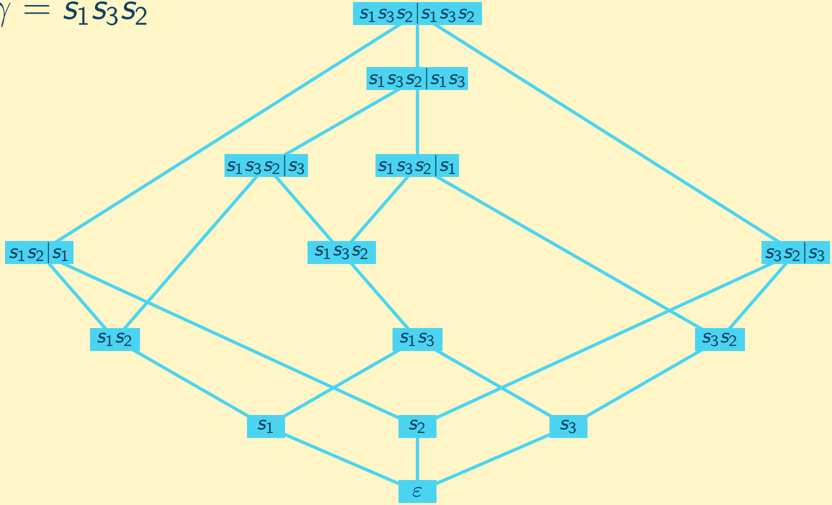

Cambrian Lattices – Example

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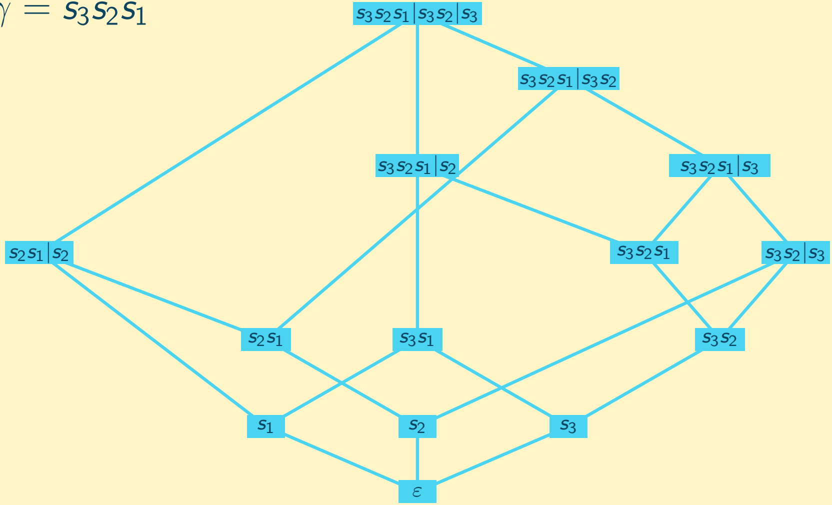

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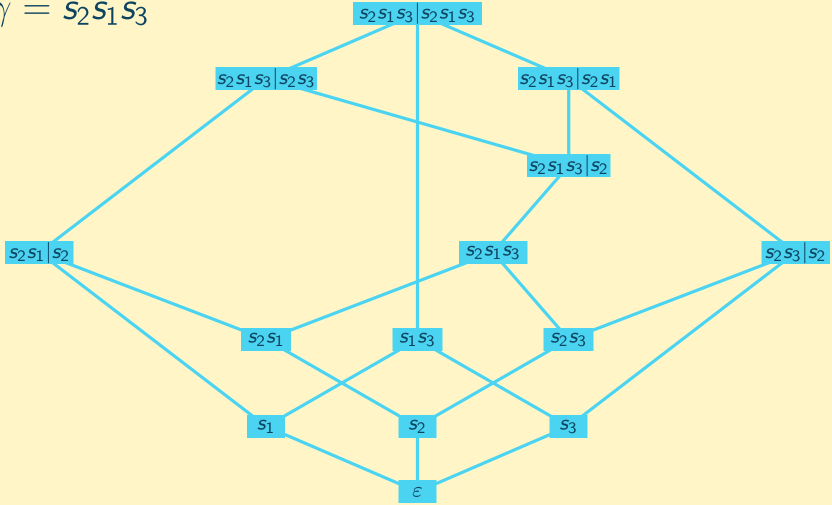

Basics on Posets

- bounded poset: a poset that has a unique minimal and a unique maximal element
- let $\mathbb{P} = (P, \leq_{\mathbb{P}})$ be a bounded poset
- $\overline{\mathbb{P}}$ is the poset that arises from $\mathbb{P}$ by removing the maximal and minimal element (the so-called proper part of $\mathbb{P}$)
- chain: linearly ordered subset c of P
  notation: $c : p_0 <_{\mathbb{P}} p_1 <_{\mathbb{P}} \cdots <_{\mathbb{P}} p_s$
- maximal chain in $[p, q]$: there is no $p' \in [p, q]$ and no $0 \leq i < s$ such that
  $p = p_0 <_{\mathbb{P}} p_1 <_{\mathbb{P}} \cdots <_{\mathbb{P}} p_i <_{\mathbb{P}} p' <_{\mathbb{P}} p_{i+1} <_{\mathbb{P}} \cdots <_{\mathbb{P}} p_s = q$
  is a chain

Edge-Labelings

- cover relation $p \lessdot_{\mathbb{P}} q$: $p <_{\mathbb{P}} q$ and there is no $p' \in P$ with $p <_{\mathbb{P}} p' <_{\mathbb{P}} q$
- $\mathcal{E}(\mathbb{P}) = \big\{(p,q) \mid p \lessdot_{\mathbb{P}} q\big\}$ is the set of covering relations on $\mathbb{P}$
- edge-labeling λ: map $\lambda : \mathcal{E}(\mathbb{P}) \to \Lambda$, for some poset Λ
- $\lambda(c) = \big(\lambda(p_0,p_1), \lambda(p_1,p_2), \ldots, \lambda(p_{s-1},p_s)\big)$ is the label-sequence of c
- rising chain: a chain c such that $\lambda(c)$ is strictly increasing
- ER-labeling: an edge-labeling such that for every interval of $\mathbb{P}$ there is exactly one rising maximal chain
- EL-labeling: an ER-labeling such that the rising chain in every interval is lexicographically first among all maximal chains

EL-Shellability

- EL-shellable poset: a bounded poset that admits an EL-labeling

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- the order complex $\Delta(\overline{\mathbb{P}})$ of an EL-shellable poset $\mathbb{P}$ is shellable and hence Cohen-Macaulay
- the geometric realization of $\Delta(\overline{\mathbb{P}})$ is homotopy equivalent to a wedge of spheres
- the i-th Betti number of $\Delta(\overline{\mathbb{P}})$ is given by the number of falling maximal chains of length $i+2$
- hence, the Euler characteristic $\chi\big(\Delta(\overline{\mathbb{P}})\big)$ can be computed from the labeling
- if $0_{\mathbb{P}}$ is the unique minimal element and $1_{\mathbb{P}}$ the unique maximal element of $\mathbb{P}$, we have $\chi\big(\Delta(\overline{\mathbb{P}})\big) = \mu(0_{\mathbb{P}}, 1_{\mathbb{P}})$

Outline

The Labeling

- recall that we write the γ-sorting word of $w \in W$ as

$$w = s_1^{\delta_{1,1}} s_2^{\delta_{1,2}} \cdots s_n^{\delta_{1,n}} | s_1^{\delta_{2,1}} s_2^{\delta_{2,2}} \cdots s_n^{\delta_{2,n}} | \cdots | s_1^{\delta_{l,1}} s_2^{\delta_{l,2}} \cdots s_n^{\delta_{l,n}},$$

  where $\delta_{i,j} \in \{0,1\}$ for $1 \leq i \leq l$ and $1 \leq j \leq n$
- define the set of filled positions of w in γ^∞ by

$$\alpha(w) = \big\{(i-1)\cdot n + j \mid \delta_{i,j} = 1\big\} \subseteq \mathbb{N}$$

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  $w = s_1^1 s_2^0 s_3^1 s_4^0 | s_1^0 s_2^1 s_3^0 s_4^1 | s_1^0 s_2^0 s_3^1 s_4^0,$

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  $w = \mathbf{s_1^1} s_2^0 s_3^1 s_4^0 | s_1^0 s_2^1 s_3^0 s_4^1 | s_1^0 s_2^0 s_3^1 s_4^0, \quad \alpha(w) = \{1\}$

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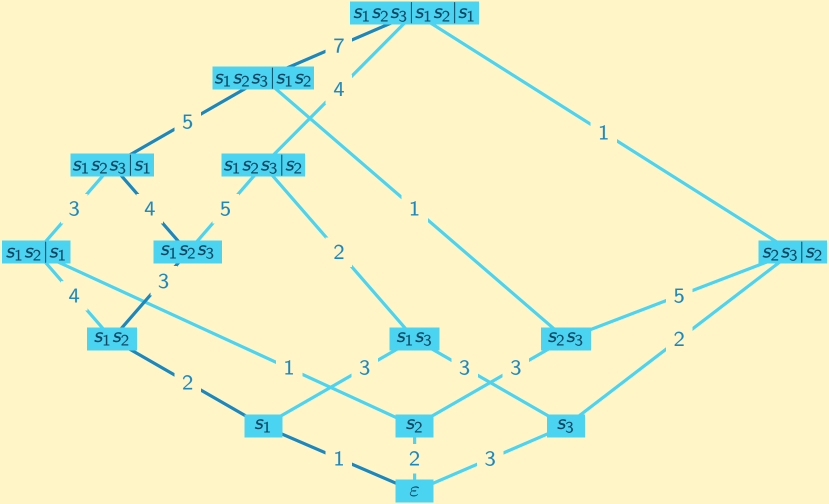
- $\lambda : \mathcal{E}(C_\gamma) \to \mathbb{N}, \quad (u,v) \mapsto \min\{\alpha(v) \setminus \alpha(u)\}$

The Labeling – Example



Main Result

Theorem

For every finite Coxeter group W and every Coxeter element $\gamma \in W$, the edge-labeling λ is an EL-labeling of C_γ.

We need two technical lemmas for the proof!

Lemma 1

Lemma

Let $u \leq v$ in C_γ. If u and v have the same first block b, then let u', v' be the elements obtained by omitting b. Then, $u', v' \in C_\gamma$, and we have:

1. *The intervals $[u, v]$ and $[u', v']$ are isomorphic.*
2. *For every $w_1', w_2' \in [u', v']$ with $w_1' \lessdot w_2'$ we have $\lambda(bw_1', bw_2') = \lambda(w_1', w_2') + n$.*

Lemma 2

Lemma

For $u, v \in C_\gamma$ with $u \leq v$ define $i_0 = \min\{i \in \alpha(v) \setminus \alpha(u)\}$. The following hold:

1. *The label i_0 appears in every maximal chain of $[u, v]$.*
2. *There is a unique element $u_1 \in (u, v)$ with $u \lessdot u_1$ and $\lambda(u, u_1) = i_0$.*
3. *$\alpha(u)$ is a subset of $\alpha(v)$.*
4. *The labels of each maximal chain in $[u, v]$ are distinct.*

Main Result

Theorem

For every finite Coxeter group W and every Coxeter element $\gamma \in W$, the edge-labeling λ is an EL-labeling of C_γ.

Sketch of proof:

- proceed by induction on the length k of the interval $[u, v]$
- if $k = 2$, then the result follows from Lemma 2
- Lemma 2 tells us that there exists an $u \lessdot u_1$ in $[u, v]$ with $\lambda(u, u_1) = i_0$
- apply induction on the interval $[u_1, v]$ to find the maximal chain $u_1 \lessdot u_2 \lessdot \cdots \lessdot v$ which is rising and lexicographically first
- by definition and Lemma 2, the chain $u \lessdot u_1 \lessdot u_2 \lessdot \cdots \lessdot v$ is the desired maximal chain in $[u, v]$

Outline

Topology of C_γ

Theorem (Reading, 2004)

Every open interval in a Cambrian lattice is either contractible or homotopy equivalent to a sphere.

- Nathan Reading obtained this result by showing that C_γ is a special instance of a fan lattice associated to a central hyperplane arrangement
- he showed this property for this larger class of lattices
- having an EL-labeling of C_γ, we can proof this property directly

Topology of C_γ

Theorem

Let $u, v \in C_\gamma$ with $u \leq v$. Then $|\mu(u, v)| \leq 1$.

- if $\mathbb{P}$ is an EL-shellable poset, and $p, q \in \mathbb{P}$ with $p \leq q$, then

$$\mu(p, q) = \#\text{ even length falling chains in } [p, q] - \#\text{ odd length falling chains in } [p, q]$$

- we show that there exists at most one falling chain in each interval

Subword Complexes

- Vincent Pilaud and Christian Stump have recently shown that the Cambrian lattices coincide with the poset of flips of special subword complexes
- Christian Stump observed that our labeling is a specialization of a natural labeling of the poset of flips for every subword complex

Thank You.

An EL-Labeling for Trim Lattices

- let L be a lattice
- left-modular element: $x \in L$ such that for all $y, z \in L$ holds

$$(y \vee_L x) \wedge_L z = y \vee_L (x \wedge_L z)$$

- left-modular lattice: a lattice that contains a maximal chain of left-modular elements
- join-irreducible element: $x \in L$ which covers exactly one element
- meet-irreducible element: $x \in L$ which is covered by exactly one element
- trim lattice: a left-modular lattice (with left-modular chain of length n) that has exactly n join- and n meet-irreducible elements

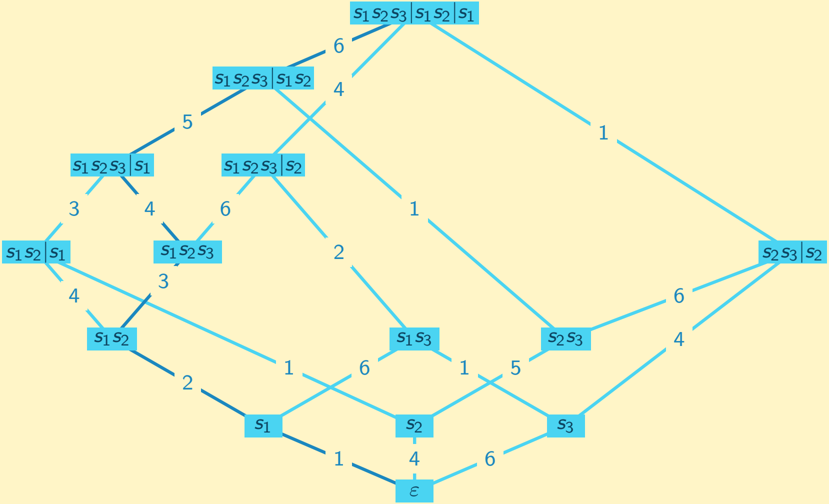
An EL-Labeling for Trim Lattices

- let L be a finite lattice with left-modular chain $\hat{0} = x_0 \lessdot_L x_1 \lessdot_L \cdots \lessdot_L x_n = \hat{1}$
- $\gamma : \mathcal{E}(L) \to \mathbb{N}, \quad (p, q) \mapsto \min\{i \mid p \vee_L x_i \wedge_L q = q\}$

Proposition (Liu, 1999)

If L is a finite, left-modular lattice, then γ is an EL-labeling.

Liu's Labeling



Our Labeling