SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_G(d,d,n)$ The Groups G(d,d,n)Perttion Lattices of G(d,d,n)A First Discomposition A Second Symmetric Chain Decompositions and the Strong Sperner Property for Noncrossing Partition Lattices

Henri Mühle

LIX (École Polytechnique)

April 05, 2016

76th Séminaire Lotharingien de Combinatoire, Ottrott

Sperner's Theorem

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decompositions

SCD of $\mathcal{NC}_G(d,d,n)$. The Groups G(d,d,n). Noncrossing Partition Lattices of G(d,d,n). A Viral Decomposition. A Second

- $[n] = \{1, 2, ..., n\}$ for $n \in \mathbb{N}$
- antichain: set of pairwise incomparable subsets of [*n*]

Theorem (E. Sperner, 1928)

The maximal size of an antichain of [n] *is* $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Sperner's Theorem

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_G(d, d, n)$ The Groupe G(d, d, n)Noncrossing Partition Lattices of (d, d, n)A First Decomposition A Second • *k*-family: family of subsets of [*n*] that can be written as a union of at most *k* antichains

Theorem (P. Erdős, 1945)

The maximal size of a k-family of [n] is the sum of the k largest binomial coefficients.

A Generalization

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decompositions

SCD of $\mathcal{NC}_{G}(d,d,n)$ The Groupe G(d,d Noncrossing Partition Lattices e G(d,d,n)A First Decomposition

Decompositio A Second Decompositio

• poset perspective:

- antichain of $[n] \longleftrightarrow$ antichain in the Boolean lattice \mathcal{B}_n
- binomial coefficients \leftrightarrow rank numbers of \mathcal{B}_n

- \mathcal{P} .. graded poset of rank n
- *k*-Sperner: size of a *k*-family does not exceed sum of *k* largest rank numbers
- strongly Sperner: k-Sperner for all $k \le n$

A Generalization

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decompositions

SCD of $\mathcal{NC}_G(d,d,n)$ The Compt G(d,d,n)Intercosing Partition Lattices of G(d,d,n)A First Decomposition A Second

- poset perspective:
 - antichain of $[n] \longleftrightarrow$ antichain in the Boolean lattice \mathcal{B}_n
 - binomial coefficients \leftrightarrow rank numbers of \mathcal{B}_n

- \mathcal{P} .. graded poset of rank n
- *k*-Sperner: size of a *k*-family does not exceed sum of *k* largest rank numbers
- **strongly Sperner**: *k*-Sperner for all $k \le n$

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decompositions

SCD of $\mathcal{NC}_{G(d,d,n)}$ The Groups G(d)Noncrossing Partition Lattice

G(d, d, n)

Decompositio

A Second Decomposition

• a strongly Sperner poset



SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_{G(d,d,n)}$ The Groups G(a)Noncrossing

G(d, d, n)

Decompositio

A Second Decomposition

• a Sperner poset that is not 2-Sperner



SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_{G(d,d,n)}$ The Groups G(a)Noncrossing

G(d, d, n)

Decompositio

A Second Decomposition

• a Sperner poset that is not 2-Sperner



SCD and SSF for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_{G(d,d,n)}$ The Groups G(d)Noncrossing Partition Lattice

A First Decomposition

A Second Decomposition

• a 2-Sperner poset that is not Sperner



SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_{G(d,d,n)}$ The Groups G(d)Noncrossing Partition Lattice

A First Decompositio

A Second Decomposition

• a 2-Sperner poset that is not Sperner



SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decompositions

SCD of $\mathcal{NC}_G(d,d,n)$ The Groupe G(d, Noncrossing Partition Lattices G(d,d,n)A First Decomposition

A Second Decomposition

• strongly Sperner posets:

- Boolean lattices
- divisor lattices
- lattices of noncrossing set partitions
- Bruhat posets of finite Coxeter groups
- weak order lattice of H_3
- non-Sperner posets:
 - lattices of set partitions
 - geometric lattices

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decompositions

SCD of $\mathcal{NC}_G(d,d,n)$ The Groups G(d,Noncrossing Partition Lattices G(d,d,n)A First Decomposition

A Second Decomposition

• strongly Sperner posets:

- Boolean lattices
- divisor lattices

- (symmetric chain decompositions)
- lattices of noncrossing set partitions
- Bruhat posets of finite Coxeter groups
- weak order lattice of *H*₃
- non-Sperner posets:
 - lattices of set partitions
 - geometric lattices

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decompositions

SCD of $\mathcal{NC}_G(d,d,n)$ The Groups G(d,Noncrossing Partition Lattices G(d,d,n)A First Decomposition

A Second Decomposition

• strongly Sperner posets:

- Boolean lattices
- divisor lattices

- (symmetric chain decompositions)
- lattices of noncrossing set partitions
- Bruhat posets of finite Coxeter groups
- weak order lattice of H_3 (no symmetric chain decomposition)

• non-Sperner posets:

- lattices of set partitions
- geometric lattices

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decompositions

SCD of $\mathcal{NC}_{G}(d,d,n)$ The Groups G(d, n)Noncrossing Partition Lattices G(d,d,n)A First Decomposition

A Second Decomposition

• strongly Sperner posets:

- Boolean lattices
- divisor lattices

- (symmetric chain decompositions)
- lattices of noncrossing set partitions
- Bruhat posets of finite Coxeter groups
- weak order lattice of H_3 (no symmetric chain decomposition)

• non-Sperner posets:

• lattices of set partitions

(of very large sets ...)

• geometric lattices

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decompositions

SCD of $\mathcal{NC}_G(d,d,n)$ The Groups G(d,Noncrossing Partition Lattices G(d,d,n)A First Decomposition

A Second Decomposition

• strongly Sperner posets:

- Boolean lattices
- divisor lattices

(symmetric chain decompositions)

- lattices of noncrossing set partitions
- Bruhat posets of finite Coxeter groups
- weak order lattice of H_3 (no symmetric chain decomposition)

• non-Sperner posets:

- lattices of set partitions
- geometric lattices

(of very large sets...) (certain bond lattices of graphs)



SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_{G}(d,d,t)$ The Groups G(d,d)Nencrossing Partition Lattices G(d,d,n)A First Decomposition A Second

Motivation

2 Symmetric Chain Decompositions



- 3 Symmetric Chain Decompositions of $\mathcal{NC}_{G(d,d,n)}$
 - The Groups G(d, d, n)
 - Noncrossing Partition Lattices of *G*(*d*, *d*, *n*)
 - A First Decomposition
 - A Second Decomposition



SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_{G}(d,d,n)$ The Groups G(d,n)Partition Lattices G(d,n)A First Decomposition A Second

1 Motivatio

2 Symmetric Chain Decompositions



Symmetric Chain Decompositions of $\mathcal{NC}_{G(d,d,n)}$

- The Groups *G*(*d*,*d*,*n*)
- Noncrossing Partition Lattices of *G*(*d*,*d*,*n*)
- A First Decomposition
- A Second Decomposition

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decompositions

SCD of $\mathcal{NC}_G(d,d,n)$ The Groupe G(d,d,n)Partition Lattices of G(d,d,n)A First Decomposition A Second • \mathcal{P} .. graded poset of rank n

• decomposition



8 / 27

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decompositions

SCD of $\mathcal{NC}_G(d,d,n)$ The Groups G(d,d,n)Portition Lattices of G(d,d,n)A First Decomposition A Second • \mathcal{P} .. graded poset of rank n



SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decompositions

SCD of $\mathcal{NC}_G(d, d, n)$ The Groups G(d, d, Noncrossing Partition Lattices to G(d, d, n)A First Decomposition $\frac{h}{2}$ Second • \mathcal{P} .. graded poset of rank n



SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decompositions

SCD of $\mathcal{NC}_G(d, d, n)$ The Groupe G(d, d, Noncreasing Partition Lattices of G(d, d, n)A First Decomposition A Second • \mathcal{P} .. graded poset of rank n



SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decompositions

SCD of $\mathcal{NC}_G(d,d,n)$ The Groupe G(d,d,n)Noncreasing Partition Lattices o G(d,d,n)A First Decomposition A Second • \mathcal{P} .. graded poset of rank n



SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decompositions

SCD of $\mathcal{NC}_G(d,d,n)$ The Groupe G(d,d,n)Noncreasing Partition Lattices o G(d,d,n)A First Decomposition A Second • \mathcal{P} .. graded poset of rank n



SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decompositions

SCD of $\mathcal{NC}_G(d,d,n)$ The Groups G(d,d,Noncreasing Partition Lattices of G(d,d,n)A First Decomposition A Second

A Second Decomposition • \mathcal{P} .. graded poset of rank n

• symmetric chain decomposition



SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decompositions

SCD of $\mathcal{NC}_{G}(d_{s}d_{s}n)$ The Groups $G(d_{s}d_{s})$ Partition Lattices of $G(d_{s}d_{s})$ A First Decomposition Decomposition

• \mathcal{P} .. graded poset of rank *n*

Theorem (Folklore)

If \mathcal{P} admits a symmetric chain decomposition, then \mathcal{P} is strongly Sperner.

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decompositions

SCD of $\mathcal{NC}_{G}(d,d,n)$ The Groups G(d,d,n)Partition Lattices of G(d,d,n)A First Decomposition Decomposition

• \mathcal{P} .. graded poset of rank *n*

Theorem (Folklore)

If \mathcal{P} and \mathcal{Q} admit a symmetric chain decomposition, then so does $\mathcal{P} \times \mathcal{Q}$.

for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_{G}(d, d, r)$ The Groups G(d, d, Noncrossing Parentor Lattices of G(d, d, r) A First Decomposition A Second Decomposition

- \mathcal{P} .. graded poset of rank *n*; N_i .. size of *i*th rank
- rank-symmetric: $N_i = N_{n-i}$
- rank-unimodal: $N_0 \leq \cdots \leq N_j \geq \cdots \geq N_n$
- **Peck**: strongly Sperner, rank-symmetric, rank-unimodal

Theorem (Folklore)

If \mathcal{P} *admits a symmetric chain decomposition, then* \mathcal{P} *is Peck.*

for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{N}C_G(d,d,n)$ The Groups G(d,d,n)Nonconstag Partner Lattices of G(d,d,n)A First Decomposition A Second Decomposition

- \mathcal{P} .. graded poset of rank n; N_i .. size of i^{th} rank
- rank-symmetric: $N_i = N_{n-i}$
- rank-unimodal: $N_0 \leq \cdots \leq N_j \geq \cdots \geq N_n$
- **Peck**: strongly Sperner, rank-symmetric, rank-unimodal

Theorem (Folklore)

If \mathcal{P} *and* \mathcal{Q} *are Peck, then so is* $\mathcal{P} \times \mathcal{Q}$ *.*



SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_{G(d,d,n)}$

The Groups G(d)Noncrossing Partition Lattice G(d, d, n)A First Decomposition

A Second Decompositio



Symmetric Chain Decompositions



- Symmetric Chain Decompositions of $\mathcal{NC}_{G(d,d,n)}$
 - The Groups G(d, d, n)
 - Noncrossing Partition Lattices of *G*(*d*, *d*, *n*)
 - A First Decomposition
 - A Second Decomposition



SCD and SSP for NCP Henri Müble

Motivation

Symmetric Chain Decom positions

SCD of NC_{G(d,d,11)} The Groups G(d,

Noncrossing Partition Lattic G(d,d,n)

A First Decomposi

A Second Decomposition



2 Symmetric Chain Decompositions



Symmetric Chain Decompositions of $\mathcal{NC}_{G(d,d,n)}$

• The Groups G(d, d, n)

• Noncrossing Partition Lattices of *G*(*d*,*d*,*n*)

- A First Decomposition
- A Second Decomposition

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_{G(d,d,n)}$

Noncrossing Partition Lattices G(d, d, n)

A First Decompositi

A Second Decomposition • G(d, d, n): group of monomial $(n \times n)$ -matrices, where

- non-zero entries are *d*th roots of unity
- product of non-zero entries is 1

$$n = 5, d = 1$$

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_{G(d,d,n)}$

Noncrossing Partition Lattices G(d, d, u)

A First Decompositi

A Second Decompositio

• G(d, d, n): group of monomial $(n \times n)$ -matrices, where

- non-zero entries are *d*th roots of unity
- product of non-zero entries is 1

n=5, d=1

Nope!
$$\rightarrow$$
 $\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_{G(d,d,n)}$

Noncrossing Partition Lattices G(d, d, n)

A First Decompositi

A Second Decomposition • G(d, d, n): group of monomial $(n \times n)$ -matrices, where

- non-zero entries are *d*th roots of unity
- product of non-zero entries is 1

$$n = 5, d = 1$$

$$\left(\begin{array}{ccccc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right)$$

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_{G(d,d,n)}$

Noncrossing Partition Lattices G(d, d, n)

A First Decompositi

A Second Decomposition • G(d, d, n): group of monomial $(n \times n)$ -matrices, where

• non-zero entries are *d*th roots of unity

• product of non-zero entries is 1

n = 5, d = 1

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \leftarrow Nope!$$

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_{G(d,d,n)}$

Noncrossing Partition Lattices G(d, d, n)

A First Decompositi

A Second Decomposition • G(d, d, n): group of monomial $(n \times n)$ -matrices, where

- non-zero entries are *d*th roots of unity
- product of non-zero entries is 1

$$n = 5, d = 1$$

$$\left(\begin{array}{ccccc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right)$$

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_{G(d,d,n)}$

Noncrossing Partition Lattices G(d,d,n)

A First Decompositi

A Second Decomposition • G(d, d, n): group of monomial $(n \times n)$ -matrices, where

- non-zero entries are *d*th roots of unity
- product of non-zero entries is 1

$$n = 5, d = 2$$
SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of NC_{G(d,d,n)}

The Groups G(d, d, n)

Noncrossing Partition Lattices G(d, d, n)

A First Decompositi

A Second Decomposition • G(d, d, n): group of monomial $(n \times n)$ -matrices, where

• non-zero entries are *d*th roots of unity

• product of non-zero entries is 1

n = 5, d = 2

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \leftarrow Nope!$$

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_{G(d,d,n)}$

Noncrossing Partition Lattices G(d,d,n)

A First Decompositi

A Second Decomposition • G(d, d, n): group of monomial $(n \times n)$ -matrices, where

- non-zero entries are *d*th roots of unity
- product of non-zero entries is 1

$$n = 5, d = 2$$

$$\left(\begin{array}{ccccc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right)$$

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decompositions

SCD of $\mathcal{NC}_{G(d,d,n)}$ The Groups G(d,d,

Noncrossing Partition Lattices cG(d, d, u)A First Decomposition

A Second Decomposition • G(d, d, n): group of monomial $(n \times n)$ -matrices, where

- non-zero entries are *d*th roots of unity
- product of non-zero entries is 1

Observation

For $n \ge 1$, the group G(1,1,n) is isomorphic to the symmetric group \mathfrak{S}_n .

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decompositions

SCD of $\mathcal{NC}_{G(d,d,n)}$ The Groups G(d,d,d)

Noncrossing Partition Lattices G(d,d,n)

Decompositio

A Second Decomposition • G(d, d, n): group of monomial $(n \times n)$ -matrices, where

- non-zero entries are *d*th roots of unity
- product of non-zero entries is 1

Observation

For $d, n \ge 1$ the group G(d, d, n) is a normal subgroup of index d in $\mu_d \wr \mathfrak{S}_n$, where μ_d is the cyclic group of d^{th} roots of unity.

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_{G(d,d,n)}$ The Groups G(d,d)

Noncrossing Partition Lattice G(d,d,u)

A First Decomposit

A Second Decomposition • subgroups of \mathfrak{S}_{dn} , permuting elements of $\{1^{(0)}, \dots, n^{(0)}, 1^{(1)}, \dots, n^{(1)}, \dots, 1^{(d-1)}, \dots, n^{(d-1)}\}$

•
$$w \in G(d, d, n)$$
 satisfies $w(k^{(s)}) = \pi(k)^{(s+t_k)}$

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_{G(d,d,n)}$ The Groups G(d,d,d)

Noncrossing Partition Lattices G(d, d, n)

A First Decomposit

A Second

subgroups of
$$\mathfrak{S}_{dn}$$
, permuting elements of $\{1^{(0)}, \dots, n^{(0)}, 1^{(1)}, \dots, n^{(1)}, \dots, 1^{(d-1)}, \dots, n^{(d-1)}\}$

$$n = 5, d = 2$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \pi = \begin{bmatrix} 5, 1, 3, 2, 4 \end{bmatrix}$$

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $NC_G(d,d,n)$ The Groups G(d,d)Noncressing Partition Lattices of G(d,d,n)

A First Decomposi

A Second Decompositio

• elements can be decomposed into "cycles":

$$\left(\begin{pmatrix} k_1^{(t_1)} \ \dots \ k_r^{(t_r)} \end{pmatrix} \right) = \begin{pmatrix} k_1^{(t_1)} \ \dots \ k_r^{(t_r)} \end{pmatrix} \begin{pmatrix} k_1^{(t_1+1)} \ \dots \ k_r^{(t_r+1)} \end{pmatrix}$$
$$\cdots \begin{pmatrix} k_1^{(t_1+d-1)} \ \dots \ k_r^{(t_r+d-1)} \end{pmatrix},$$

and

$$\begin{bmatrix} k_1^{(t_1)} \ \dots \ k_r^{(t_r)} \end{bmatrix}_s = \begin{pmatrix} k_1^{(t_1)} \ \dots \ k_r^{(t_r)} \ k_1^{(t_1+s)} \ \dots \\ k_r^{(t_r+s)} \ \dots \ k_1^{(t_1(d-1)s)} \ \dots \ k_r^{(t_r+(d-1)s)} \end{pmatrix}.$$

SCD and SSF for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_{G(d,d,n)}$ The Groups G(d,d)

Noncrossing Partition Lattice G(d,d,n)

A First Decomposi

A Second Decompositio n = 5, d = 2

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \left(\left(1^{(0)} \ 5^{(0)} \ 4^{(1)} \ 2^{(0)} \right) \right)$$



A Second Decomposition

Observation

For $d, n \ge 1$ the group G(d, d, n) is generated by

$$T = \left\{ \left(\left(i^{(0)} \, j^{(s)} \right) \right) \mid 1 \le i < j \le n, 0 \le s < d \right\}.$$

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decorr positions

SCD of $\mathcal{NC}_{G(d,d,rt)}$ The Groups G(d,d,r)Noncrossing Partition Lattices of

A First Decomposition A Second • ℓ_T .. minimal length of decomposition into elements of T

Observation

For $d, n \ge 1$ the group G(d, d, n) is generated by

$$T = \left\{ \left(\left(i^{(0)} \; j^{(s)} \right) \right) \; | \; 1 \le i < j \le n, 0 \le s < d \right\}.$$

A Partial Order

SCD and SSF for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_{G(d,d,n)}$

Noncrossing Partition Lattices

A First

A Second Decomposition

• **absolute order**: $u \leq_T v$ if and only if $\ell_T(v) = \ell_T(u) + \ell_T(u^{-1}v)$

A Partial Order

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_{G(d,d,t)}$ The Groups G(d,d)

Partition Lattices G(d, d, u)

A First Decomposition

A Second Decomposition

• **absolute order**: $u \leq_T v$ if and only if $\ell_T(v) = \ell_T(u) + \ell_T(u^{-1}v)$





SCD and SSP for NCP Henri Müble

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_{G(d,d,n)}$

Noncrossing Partition Lattices of C(d, d, w)

A First Decompositio

A Second Decompositio



2 Symmetric Chain Decompositions



Symmetric Chain Decompositions of NC_{G(d,d,n)}
 The Groups G(d, d, n)

• Noncrossing Partition Lattices of *G*(*d*, *d*, *n*)

- A First Decomposition
- A Second Decomposition

SCD and SSF for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_{G(d,d,n)}$

The Groups G(d, d,:

Noncrossing Partition Lattices G(d, d, n)

A First Decompositio

•
$$\mathcal{NC}_{G(1,1,n)}$$
: interval $[e,c]_T$ in $(G(1,1,n), \leq_T)$ for $c = (1 \ 2 \ \dots \ n)$

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of NC_{G(d,d,n)}

Noncrossing Partition Lattices G(d, d, n)

A First Decomposition





SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_{G(d,d,n)}$

The Groups G(d, d)

Noncrossing Partition Lattices G(d, d, n)

A First Decompositi





SCD and SSF for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_{G(d,d,n)}$

The Groups G(d, d,

Noncrossing Partition Lattices G(d,d,n)

A First Decompositio

•
$$\mathcal{NC}_{G(d,d,n)}$$
: interval $[e, \gamma]_T$ in $(G(d, d, n), \leq_T)$ for
 $\gamma = \left[1^{(0)} 2^{(0)} \dots (n-1)^{(0)}\right]_1 \left[n^{(0)}\right]_{d-1}$

for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of NC_{G(d,d,n)}

Noncrossing Partition Lattices G(d, d, n)

A First Decomposition A Second Decomposition



SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_{G(d,d,n)}$

Noncrossing Partition Lattices G(d, d, n)

A First Decompositi

A Second Decomposition



17 / 27

Symmetric Chain Decompositions of $\mathcal{NC}_{G(1,1,n)}$



Symmetric Chain Decompositions of $\mathcal{NC}_{G(1,1,n)}$

for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of *NC_{G(d,d,n})*

The Groups G(d, d, i)

Noncrossing Partition Lattices CG(d, d, n)

A First Decomposition

A Second Decomposition

- $R_k = \{ w \leq_T c \mid w(1) = k \}, \mathcal{R}_k = (R_k, \leq_T)$
- 🗄 .. disjoint set union; **2** .. 2-chain

Theorem (R. Simion & D. Ullmann, 1991)

The lattice $\mathcal{NC}_{G(1,1,n)}$ *admits a symmetric chain decomposition for each* $n \geq 1$ *.*

Symmetric Chain Decompositions of $\mathcal{NC}_{G(1,1,n)}$

for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_{G(d,d,n)}$

The Groups G(d,d, 1

Noncrossing Partition Lattices CG(d, d, n)

A First Decomposition

Decomposition

• $R_k = \{ w \leq_T c \mid w(1) = k \}, \mathcal{R}_k = (R_k, \leq_T)$

• 🗄 .. disjoint set union; **2** .. 2-chain

Lemma (R. Simion & D. Ullmann, 1991)

We have $\mathcal{R}_1 \uplus \mathcal{R}_2 \cong \mathbf{2} \times \mathcal{NC}_{G(1,1,n-1)}$, and $\mathcal{R}_i \cong \mathcal{NC}_{G(1,1,i-2)} \times \mathcal{NC}_{G(1,1,n-i+1)}$ whenever $3 \le i \le n$. Moreover, this decomposition is symmetric.

Example: $\mathcal{NC}_{G(1,1,4)}$



Example: $\mathcal{NC}_{G(1,1,4)}$





SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_G(d,d,n)$ The Groups G(d,c)Noncrossing Partition Lattices G(d,d,n)

A First Decomposit

A Second Decomposition



2 Symmetric Chain Decompositions



Symmetric Chain Decompositions of $\mathcal{NC}_{G(d,d,n)}$

- The Groups *G*(*d*,*d*,*n*)
- Noncrossing Partition Lattices of *G*(*d*,*d*,*n*)
- A First Decomposition
- A Second Decomposition

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decorr positions

SCD of $\mathcal{NC}_{G(d,d,n)}$ The Groups G(d,d,Noncrossing Partition Lattices o G(d,d,n)

A First Decompositio

•
$$R_k^{(s)} = \left\{ w \leq_T \gamma \mid w(1^{(0)}) = k^{(s)} \right\}$$

• $\mathcal{R}_k^{(s)} = \left(R_k^{(s)}, \leq_T \right)$

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_{G(d,d,n)}$ The Groups G(d,d,Noncrossing Partition Lattices of G(d,d,n)

A First Decompositio

A Second Decomposition

•
$$R_k^{(s)} = \left\{ w \leq_T \gamma \mid w(1^{(0)}) = k^{(s)} \right\}$$

• $\mathcal{R}_k^{(s)} = \left(R_k^{(s)}, \leq_T \right)$

Lemma (**%**, 2015)

The sets $R_1^{(s)}$ and $R_k^{(s')}$ are empty for $2 \le s < d$ as well as $2 \le k < n$ and $1 \le s' < d - 1$.

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_{G(d,d,n)}$ The Groups G(d,d,Noncrossing Partition Lattices of G(d,d,n)

A First Decomposition

A Second Decomposition

•
$$R_k^{(s)} = \left\{ w \leq_T \gamma \mid w(1^{(0)}) = k^{(s)} \right\}$$

• $\mathcal{R}_k^{(s)} = \left(R_k^{(s)}, \leq_T \right)$

Lemma (**%**, 2015)

The poset $\mathcal{R}_1^{(0)} \uplus \mathcal{R}_2^{(0)}$ is isomorphic to $\mathbf{2} \times \mathcal{NC}_{G(d,d,n-1)}$. Moreover, its least element has length 0, and its greatest element has length n.

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_{G(d,d,n)}$ The Groups G(d,d,Noncrossing Partition Lattices of G(d,d,n)

A First Decomposition

A Second Decomposition

•
$$R_k^{(s)} = \left\{ w \leq_T \gamma \mid w(1^{(0)}) = k^{(s)} \right\}$$

• $\mathcal{R}_k^{(s)} = \left(R_k^{(s)}, \leq_T \right)$

Lemma (**%**, 2015)

The poset $\mathcal{R}_n^{(s)}$ is isomorphic to $\mathcal{NC}_{G(1,1,n-1)}$ for $0 \le s < d$. Moreover, its least element has length 1, and its greatest element has length n - 1.

for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_{G(d,d,n)}$ The Groups G(d,d,Noncrossing Partition Lattices of G(d,d,n)

A First Decompositio

A Second Decomposition

•
$$R_k^{(s)} = \left\{ w \leq_T \gamma \mid w(1^{(0)}) = k^{(s)} \right\}$$

• $\mathcal{R}_k^{(s)} = \left(R_k^{(s)}, \leq_T \right)$

Lemma (**%**, 2015)

The poset $\mathcal{R}_i^{(0)}$ is isomorphic to $\mathcal{NC}_{G(d,d,n-i+1)} \times \mathcal{NC}_{G(1,1,i-2)}$ whenever $3 \leq i < n$. Moreover, its least element has length 1, and its greatest element has length n - 1.

for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_{G(d,d,n)}$ The Groups G(d,d,Noncrossing Partition Lattices of G(d,d,n)

A First Decomposition

A Second Decomposition

•
$$R_k^{(s)} = \left\{ w \leq_T \gamma \mid w(1^{(0)}) = k^{(s)} \right\}$$

• $\mathcal{R}_k^{(s)} = \left(R_k^{(s)}, \leq_T \right)$

Lemma (**%**, 2015)

The poset $\mathcal{R}_i^{(d-1)}$ is isomorphic to $\mathcal{NC}_{G(1,1,n-i)} \times \mathcal{NC}_{G(d,d,i-1)}$ whenever $3 \leq i < n$. Moreover, its least element has length 1, and its greatest element has length n - 1.

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decorr positions

SCD of $\mathcal{NC}_{G(d,d,n)}$ The Groups G(d,d,n)Noncrossing Partition Lattices o G(d,d,n)

A First Decomposition

A Second Decomposition

•
$$R_k^{(s)} = \left\{ w \leq_T \gamma \mid w(1^{(0)}) = k^{(s)} \right\}$$

• $\mathcal{R}_k^{(s)} = \left(R_k^{(s)}, \leq_T \right)$

Lemma (**%**, 2015)

The poset $\mathcal{R}_1^{(1)}$ is isomorphic to $\mathcal{NC}_{G(1,1,n-2)}$. Moreover, its least element has length 2, and its greatest element has length n-1.

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decorr positions

SCD of $\mathcal{NC}_{G(d,d,n)}$ The Groups G(d,d,Noncrossing Partition Lattices of G(d,d,n)

A First Decompositio

A Second Decomposition

•
$$R_k^{(s)} = \left\{ w \leq_T \gamma \mid w(1^{(0)}) = k^{(s)} \right\}$$

• $\mathcal{R}_k^{(s)} = \left(R_k^{(s)}, \leq_T \right)$

Lemma (**%**, 2015)

The poset $\mathcal{R}_2^{(d-1)}$ is isomorphic to $\mathcal{NC}_{G(1,1,n-2)}$. Moreover, its least element has length 1, and its greatest element has length n-2.

Example: $\mathcal{NC}_{G(3,3,3)}$



Example: $\mathcal{NC}_{G(3,3,3)}$





SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_G(d,d,n)$ The Groups G(d, Noncrossing Partition Lattices G(d,d,n)A First Decomposition

A Second Decomposition



2 Symmetric Chain Decompositions



Symmetric Chain Decompositions of $\mathcal{NC}_{G(d,d,n)}$

- The Groups *G*(*d*,*d*,*n*)
- Noncrossing Partition Lattices of *G*(*d*,*d*,*n*)
- A First Decomposition
- A Second Decomposition
SCD and SSF for NCP Henri Mühle

• bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decompositions

SCD of $\mathcal{NC}_G(d,d,n)$ The Groupe G(d,d,n)Partition Lattices of G(d,d,n)A First Decomposition

A Second Decomposition • bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$

• consider the map

$$f_1: R_1^{(1)} \to N\!C_{G(d,d,n)}(\gamma), \quad x \mapsto \left(\left(1^{(0)} \ n^{(d-2)} \right) \right) x$$

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decompositions

SCD of $\mathcal{NC}_G(d,d,n)$ The Groups G(d,d,Noncrossing Partition Lattices of G(d,d,n)A First Decomposition

A Second Decomposition

- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$
- consider the map

$$f_1: R_1^{(1)} \to R_n^{(d-1)}, \quad x \mapsto \left(\left(1^{(0)} \ n^{(d-2)} \right) \right) x$$

for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_G(d,d,n)$ The Groups G(d,d,n)Partition Lattices of G(d,n)A first, Decomposition A Second

- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$
- consider the map

$$f_1: R_1^{(1)} \to R_n^{(d-1)}, \quad x \mapsto \left(\left(1^{(0)} \ n^{(d-2)} \right) \right) x$$

- this map is an injective involution
- its image consists of permutations $w \in R_n^{(d-1)}$ with $w(n^{(d-1)}) = 1^{(0)}$

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $NC_G(d,d,n)$ The Coroups G(d,d,n)Partition Lattices of G(d,d,n)A First Decomposition: A Second

Decomposition

- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$
- consider the map

$$f_1: R_1^{(1)} \to R_n^{(d-1)}, \quad x \mapsto \left(\left(1^{(0)} \ n^{(d-2)} \right) \right) x$$

- this map is an injective involution
- its image is the interval $\left[\left(\left(1^{(0)} n^{(d-1)}\right)\right), \left(\left(1^{(0)} n^{(d-1)}\right)\right)\left(\left(2^{(0)} \dots (n-1)^{(0)}\right)\right)\right]_T$

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decompositions

SCD of $\mathcal{NC}_{G}(d, d, n)$ The Groups G(d, d, Noncrossing Partition Lattices of G(d, d, n) A First Decomposition A Second Decomposition • bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$

• consider the map

$$f_1: R_1^{(1)} \to R_n^{(d-1)}, \quad x \mapsto \left(\left(1^{(0)} \ n^{(d-2)} \right) \right) x$$

Lemma (**%**, 2015)

The interval
$$(f_1(R_1^{(1)}), \leq_T)$$
 is isomorphic to $\mathcal{NC}_{G(1,1,n-2)}$.

for NCP Henri Mühle

Motivation

Symmetric Chain Decompositions

SCD of $\mathcal{NC}_G(d,d,n)$. The Compt G(d,d,n). Noncrossing Partition Lattices of G(d,d,n). A First Decomposition A Second

A Second Decomposition

- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$
- consider the map

$$f_1: R_1^{(1)} \to R_n^{(d-1)}, \quad x \mapsto \left(\left(1^{(0)} \ n^{(d-2)} \right) \right) x$$

• define
$$D_1 = R_1^{(1)} \uplus f_1(R_1^{(1)})$$
, and $\mathcal{D}_1 = (D_1, \leq_T)$

for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

 $\begin{array}{l} {\rm SCD} \ {\rm of} \\ {\cal NC}_G(d,d,n) \\ {\rm The Groups G(d,d,n)} \\ {\rm Pertition Lattices of} \\ (d,d,n) \\ {\rm A First} \\ {\rm Decomposition} \\ {\rm A Second} \\ {\rm Decomposition} \end{array}$

- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$
- consider the map

$$f_1: R_1^{(1)} \to R_n^{(d-1)}, \quad x \mapsto \left(\left(1^{(0)} \ n^{(d-2)} \right) \right) x$$

• define
$$D_1 = R_1^{(1)} \uplus f_1(R_1^{(1)})$$
, and $\mathcal{D}_1 = (D_1, \leq_T)$

Lemma (**¾**, 2015)

The poset D_1 is isomorphic to $\mathbf{2} \times \mathcal{NC}_{G(1,1,n-2)}$. Moreover, its least element has length 1, and its greatest element has length n-1.

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decompositions

SCD of $\mathcal{NC}_{G(d,d,n)}$ The Groups G(d,d,n)Partition Lattices c G(d,d,n)A First Decomposition

A Second Decomposition

- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$
- consider the map

$$f_2: R_2^{(d-1)} \to NC_{G(d,d,n)}(\gamma), \quad x \mapsto \left(\left(2^{(0)} \ n^{(0)} \right) \right) x$$

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decompositions

SCD of $\mathcal{NC}_G(d,d,n)$ The Groups G(d,d,n)Partition Lattices c G(d,d,n)A First Decomposition

A Second Decompositio

- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$
- consider the map

$$f_2: R_2^{(d-1)} \to R_n^{(\hat{d}-1)}, \quad x \mapsto \left(\left(2^{(0)} \ n^{(0)} \right) \right) x$$

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_G(d,d,n)$ The Groups G(d,d,n)Partition Lattices of G(d,n)A limit Decomposition A Second • bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$

• consider the map

$$f_2: R_2^{(d-1)} \to R_n^{(d-1)}, \quad x \mapsto \left(\left(2^{(0)} \ n^{(0)} \right) \right) x$$

- this map is an injective involution
- its image consists of permutations $w \in R_n^{(d-1)}$ with $w(n^{(d-1)}) = 2^{(d-1)}$

for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_{G(d,d,n)}$ The Groups G(d,d, N)Noncrossing Partition Lattices of G(d,d,n)A First Decomposition

A Second Decomposition

- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$
- consider the map

$$f_2: R_2^{(d-1)} \to R_n^{(d-1)}, \quad x \mapsto \left(\left(2^{(0)} \ n^{(0)} \right) \right) x$$

• this map is an injective involution

• its image is the interval $\left[\left(\left(1^{(0)} n^{(d-1)} 2^{(d-1)}\right)\right), \left(\left(1^{(0)} n^{(d-1)} 2^{(d-1)} \dots (n-1)^{(d-1)}\right)\right)\right]_T$

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_G(d,d,r)$. The Groups G(d,d, r), Noncreasing, Partition Lattices c G(d,d,n). A First Decomposition **A** Second • bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$

• consider the map

$$f_2: R_2^{(d-1)} \to R_n^{(d-1)}, \quad x \mapsto \left(\left(2^{(0)} \ n^{(0)} \right) \right) x$$

Lemma (**%**, 2015)

The interval
$$(f_2(R_2^{(d-1)}), \leq_T)$$
 is isomorphic to $\mathcal{NC}_{G(1,1,n-2)}$.

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_G(d,d,n)$ The Groups G(d,d,n)Partition Lattices of G(d,d,n)A First Decemposition A Second

Decomposition

- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$
- consider the map

 $f_2: R_2^{(d-1)} \to R_n^{(d-1)}, \quad x \mapsto \left(\left(2^{(0)} \ n^{(0)} \right) \right) x$

• define
$$D_2 = R_2^{(d-1)} \uplus f_2(R_2^{(d-1)})$$
, and $\mathcal{D}_2 = (D_2, \leq_T)$

SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decompositions

SCD of $\mathcal{NC}_G(d,d,n)$. The Groups G(d,d,n)Partition Lattices of G(d,d,n)A First Decomposition Decomposition

- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$
- consider the map

 $f_2: R_2^{(d-1)} \to R_n^{(d-1)}, \quad x \mapsto \left(\left(2^{(0)} \ n^{(0)} \right) \right) x$

• define
$$D_2 = R_2^{(d-1)} \uplus f_2(R_2^{(d-1)})$$
, and $\mathcal{D}_2 = (D_2, \leq_T)$

Lemma (**¾**, 2015)

The poset D_2 is isomorphic to $\mathbf{2} \times \mathcal{NC}_{G(1,1,n-2)}$. Moreover, its least element has length 1, and its greatest element has length n-1.

for NCP Henri Mühle

Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_G(d,d,n)$. The Groups G(d,d, Nencossing Partition Lattices $\mathcal{O}(d,d,n)$. A Plot Decomposition A Second Decomposition

• bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$ • define $D = R_n^{(d-1)} \setminus \left(f_1\left(R_1^{(1)}\right) \uplus f_2\left(R_2^{(d-1)}\right) \right)$, and $\mathcal{D} = (D, \leq_T)$

Lemma (**%**, 2015)

The poset \mathcal{D} is isomorphic to $\biguplus_{i=3}^{n-1} \mathcal{NC}_{G(1,1,i-2)} \times \mathcal{NC}_{G(1,1,n-i)}$. Morever, its minimal elements have length 2, and its maximal elements have length n-2.

The Main Result



Motivation

Symmetric Chain Decom positions

SCD of $\mathcal{NC}_G(d_3d_3n)$ The Groups G(d_3d_3n) Nencrossing Particle Lattrees of G(d_3d_3n) A First Decomposition A Second.

Theorem (**%**, 2015)

For $d, n \ge 2$ the lattice $\mathcal{NC}_{G(d,d,n)}$ admits a symmetric chain decomposition. Consequently, it is Peck.

Example: $\mathcal{NC}_{G(3,3,3)}$



Example: $\mathcal{NC}_{G(3,3,3)}$



SCD and SSP for NCP Henri Mühle

Motivation

Symmetric Chain Decon positions

SCD of $\mathcal{NC}_{G(d,d,n)}$

Noncrossing Partition Lattice. G(d, d, n)

A First Decompositi

A Second Decomposition

Thank You.

The General Setting

- *G*(1,1,*n*) and *G*(*d*,*d*,*n*) are well-generated irreducible complex reflection groups
- $(1 \ 2 \ \dots \ n)$ and $[1^{(0)} \ 2^{(0)} \ \dots \ (n-1)^{(0)}]_1 [n^{(0)}]_{d-1}$ are Coxeter elements
- $\mathcal{NC}_W(c)$: interval $[e, c]_T$ in (W, \leq_T) for some Coxeter element $c \in W$

The General Setting

- *G*(1,1,*n*) and *G*(*d*,*d*,*n*) are well-generated irreducible complex reflection groups
- $(1 \ 2 \ \dots \ n)$ and $[1^{(0)} \ 2^{(0)} \ \dots \ (n-1)^{(0)}]_1 [n^{(0)}]_{d-1}$ are **Coxeter elements**
- $\mathcal{NC}_W(c)$: interval $[e, c]_T$ in (W, \leq_T) for some Coxeter element $c \in W$

The General Setting

SCD and SSP for NCP Henri Mühle

- *G*(1,1,*n*) and *G*(*d*,*d*,*n*) are well-generated irreducible complex reflection groups
- $(1 \ 2 \ \dots \ n)$ and $[1^{(0)} \ 2^{(0)} \ \dots \ (n-1)^{(0)}]_1 [n^{(0)}]_{d-1}$ are Coxeter elements
- $\mathcal{NC}_W(c)$: interval $[e, c]_T$ in (W, \leq_T) for some Coxeter element $c \in W$

Theorem (¥, 2015)

The lattice \mathcal{NC}_W is Peck for any well-generated complex reflection group W.

SCD and SSF for NCP Henri Mühle

• seen: $\mathcal{NC}_{G(1,1,n)}$ and $\mathcal{NC}_{G(d,d,n)}$

- remaining: $\mathcal{NC}_{G(d,1,n)}$ and exceptional groups
- we have $\mathcal{NC}_{G(2,1,n)} \cong \mathcal{NC}_{G(d,1,n)}$ for $d \ge 2$ and $n \ge 1$

- seen: $\mathcal{NC}_{G(1,1,n)}$ and $\mathcal{NC}_{G(d,d,n)}$
- remaining: $\mathcal{NC}_{G(d,1,n)}$ and exceptional groups
- we have $\mathcal{NC}_{G(2,1,n)} \cong \mathcal{NC}_{G(d,1,n)}$ for $d \ge 2$ and $n \ge 1$

- seen: $\mathcal{NC}_{G(1,1,n)}$ and $\mathcal{NC}_{G(d,d,n)}$
- remaining: $\mathcal{NC}_{G(d,1,n)}$ and exceptional groups
- we have $\mathcal{NC}_{G(2,1,n)} \cong \mathcal{NC}_{G(d,1,n)}$ for $d \ge 2$ and $n \ge 1$

SCD and SSP for NCP Henri Mühle

- seen: $\mathcal{NC}_{G(1,1,n)}$ and $\mathcal{NC}_{G(d,d,n)}$
- remaining: $\mathcal{NC}_{G(d,1,n)}$ and exceptional groups
- we have $\mathcal{NC}_{G(2,1,n)} \cong \mathcal{NC}_{G(d,1,n)}$ for $d \ge 2$ and $n \ge 1$

Theorem (V. Reiner, 1997)

The lattice $\mathcal{NC}_{G(2,1,n)}$ *admits a symmetric chain decomposition for any* $n \ge 1$ *.*

- \mathcal{P} .. graded poset of rank n
- $\mathcal{P}[i]$.. subposet of \mathcal{P} with *i* largest ranks removed

- \mathcal{P} .. graded poset of rank n
- $\mathcal{P}[i]$.. subposet of \mathcal{P} with *i* largest ranks removed



- \mathcal{P} .. graded poset of rank n
- $\mathcal{P}[i]$.. subposet of \mathcal{P} with *i* largest ranks removed



- \mathcal{P} .. graded poset of rank n
- $\mathcal{P}[i]$.. subposet of \mathcal{P} with i largest ranks removed



- \mathcal{P} .. graded poset of rank n
- $\mathcal{P}[i]$.. subposet of \mathcal{P} with i largest ranks removed



- \mathcal{P} .. graded poset of rank n
- $\mathcal{P}[i]$.. subposet of \mathcal{P} with i largest ranks removed



- \mathcal{P} .. graded poset of rank n
- $\mathcal{P}[i]$.. subposet of \mathcal{P} with *i* largest ranks removed



- \mathcal{P} .. graded poset of rank n
- $\mathcal{P}[i]$.. subposet of \mathcal{P} with i largest ranks removed



- \mathcal{P} .. graded poset of rank n
- $\mathcal{P}[i]$.. subposet of \mathcal{P} with *i* largest ranks removed


SCD and SSP for NCP Henri Mühle

- \mathcal{P} .. graded poset of rank n
- $\mathcal{P}[i]$.. subposet of \mathcal{P} with *i* largest ranks removed

 $\mathcal{P}[4]$

SCD and SSP for NCP Henri Mühle

- \mathcal{P} .. graded poset of rank n
- $\mathcal{P}[i]$.. subposet of \mathcal{P} with *i* largest ranks removed

Proposition (¥, 2015)

A graded poset \mathcal{P} of rank n is strongly Sperner if and only if $\mathcal{P}[i]$ is Sperner for all $i \in \{0, 1, ..., n\}$.

• antichains in $\mathcal{P}[i]$ are antichains in $\mathcal{P}[s]$ for s < i

SCD and SSP for NCP Henri Mühle

- \mathcal{P} .. graded poset of rank n
- $\mathcal{P}[i]$.. subposet of \mathcal{P} with *i* largest ranks removed

Proposition (¥, 2015)

A graded poset \mathcal{P} of rank n is strongly Sperner if and only if $\mathcal{P}[i]$ is Sperner for all $i \in \{0, 1, ..., n\}$.

• antichains in $\mathcal{P}[i]$ are antichains in $\mathcal{P}[s]$ for s < i

SCD and SSP for NCP Henri Mühle

• SAGE has a fast implementation to compute the size of the largest antichain of a poset

SCD and SSP for NCP Henri Mühle

• SAGE has a fast implementation to compute the **width** of a poset

SCD and SSP for NCP Henri Mühle

• SAGE has a fast implementation to compute the **width** of a poset

Theorem (🐇, 2015)

The lattice \mathcal{NC}_W is Peck for any well-generated exceptional complex reflection group W.

SCD and SSP for NCP Henri Mühle

- *W* .. well-generated complex reflection group; *c* .. Coxeter element of *W*
- *m*-divisible noncrossing partition: *m*-multichain of noncrossing partitions → NC^(m)_W(c)

$$(w)_m = (w_1, w_2, \dots, w_m)$$
 with $w_1 \leq_T w_2 \leq_T \dots \leq_T w_m \leq_T c$

SCD and SSP for NCP Henri Mühle

- *W* .. well-generated complex reflection group; *c* .. Coxeter element of *W*
- *m*-divisible noncrossing partition: *m*-multichain of noncrossing partitions → NC^(m)_W(c)
- *m*-delta sequence: sequence of "differences" of elements in a multichain

$$(w)_m = (w_1, w_2, \dots, w_m) \text{ with } w_1 \leq_T w_2 \leq_T \dots \leq_T w_m \leq_T c$$

$$\partial(w)_m = [w_1; w_1^{-1} w_2, w_2^{-1} w_3, \dots, w_{m-1}^{-1} w_m, w_m^{-1} c]$$

SCD and SSP for NCP Henri Mühle

- *W* .. well-generated complex reflection group; *c* .. Coxeter element of *W*
- *m*-divisible noncrossing partition: *m*-multichain of noncrossing partitions → NC^(m)_W(c)
- *m*-delta sequence: sequence of "differences" of elements in a multichain
- partial order: $(u)_m \leq (v)_m$ if and only if $\partial(u)_m \leq_T \partial(v)_m \qquad \rightsquigarrow \mathcal{NC}_W^{(m)}(c)$

Question (D. Armstrong, 2009)

Are the posets $\mathcal{NC}_{W}^{(m)}$ strongly Sperner for any W and any $m \geq 1$?

SCD and SSP for NCP Henri Mühle

• affirmative answer for m = 1

Question (D. Armstrong, 2009)

Are the posets $\mathcal{NC}_{W}^{(m)}$ strongly Sperner for any W and any $m \geq 1$?

SCD and SSP for NCP Henri Mühle

- affirmative answer for m = 1
- what about m > 1?
 - $\mathcal{NC}_W^{(m)}$ is antiisomorphic to an order ideal in $(\mathcal{NC}_W)^m$
 - $(\mathcal{NC}_W)^m$ is Peck
 - *NC*^(m)_W is not rank-symmetric → no symmetric chain decomposition

Question (D. Armstrong, 2009)

Are the posets $\mathcal{NC}_{W}^{(m)}$ strongly Sperner for any W and any $m \geq 1$?



Example: $\mathcal{NC}^{(2)}_{G(1,1,4)}$

