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Combinatorics of Fuß-Catalan Posets Associated with Reflection Groups

Verfasser

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Summary

In the present thesis we investigate three different families of posets: the m-Tamari lattices, the Cambrian semilattices, and the posets of m-divisible noncrossing partitions. These families of posets have two crucial properties in common: on the one hand the cardinalities of their members are given by a generalized Catalan number, and on the other hand they arise in the context of reflection groups. We are mainly interested in topological properties of these posets, i.e. we want to understand the nature of a certain topological space associated with these posets. A very helpful combinatorial tool for this kind of investigation is a so-called EL-labeling of these posets. The existence of such a labeling implies that the associated topological space is a wedge of spheres. Moreover, we can compute the values of the Möbius function (and hence the number and the dimension of these spheres) from this labeling. For each of the considered families of posets we use a uniform labeling that does not depend on the parameters of the poset, but only on the membership of the poset in the corresponding family. Subsequently we show that this labeling has the desired properties, and we compute the values of the Möbius function.

Moreover, we investigate the m-Tamari lattices and the Cambrian semilattices from a structural point of view. It is well known that both of these families have common members, namely the Tamari lattices. Throughout the last decades the Tamari lattices have been well studied, and many nice structural and enumerative properties have been found. We prove for several of these properties that they hold analogously in both generalizations of the Tamari lattices. In particular, we provide results on the breadth, the length, the number of irreducibles and the left-modularity of these posets. Moreover, we define another labeling of the Cambrian semilattices, with which we show that these posets are bounded-homomorphic images of free lattices. We also present a first approach to unify both generalizations. The goal of this approach is to define a generalization of the Tamari lattice that is parametrized by a positive integer m and a reflection group W such that for $m = 1$ we obtain the corresponding Cambrian semilattice, and in the case where W is the symmetric group we obtain the m-Tamari lattice. For this we define the so-called m-cover poset of a given poset. Our construction works nicely for the symmetric group and for the dihedral groups, however it cannot be extended directly to the other reflection groups.

Zusammenfassung

In der vorliegenden Dissertation untersuchen wir drei verschiedene Familien von geordneten Mengen: die m-Tamariverbände, die kambrischen Halbverbände und die Ordnungen der m-teilbaren nichtkreuzenden Partitionen. Diese Familien haben zwei grundlegende Eigenschaften gemein: zum Einen sind die Kardinalitäten ihrer Elemente durch verallgemeinerte Catalanzahlen gegeben, und zum Anderen entstehen sie im Zusammenhang mit Spiegelungsgruppen. Wir sind hauptsächlich an topologischen Eigenschaften dieser Ordnungen interessiert, d.h. wir möchten die Natur eines bestimmten topologischen Raumes verstehen, der mit diesen Ordnungen fest verknüpft ist. Ein äußerst hilfreiches, kombinatorisches Werkzeug für diese Art von Untersuchung ist eine sogenannte EL-Beschriftung dieser Ordnungen. Die Existenz einer solchen Beschriftung impliziert, dass der zugehörige topologische Raum ein Sphärenkeil ist. Darüber hinaus können wir die Werte der Möbiusfunktion (und damit die Anzahl und Dimension der vorkommenden Sphären) mit Hilfe einer solchen Beschriftung berechnen. Wir verwenden für jede der betrachteten Familien von Ordnungen eine spezielle Beschriftung, die nicht von der Parametrisierung der verwendeten Ordnung abhängt, sondern nur von der Zugehörigkeit der Ordnung zu der entsprechenden Familie. Anschließend zeigen wir, dass diese Beschriftung die gewünschten Eigenschaften hat, und wir berechnen die Werte der Möbiusfunktion.

Darüber hinaus untersuchen wir die m-Tamariverbände und die kambrischen Halbverbände noch aus einem strukturellen Blickwinkel. Es ist bekannt, dass beide Familien gemeinsame Mitglieder haben, nämlich die Tamariverbände. In den letzten Jahrzehnten wurden die Tamariverbände sehr umfangreich untersucht und es wurden vielen schöne strukturelle und abzählende Eigenschaften gefunden. Wir zeigen für einige dieser Eigenschaften, dass sie analog in beiden Verallgemeinerungen der Tamariverbände gelten. Insbesondere präsentieren wir Ergebnisse über die Breite, die Länge, die Anzahl der irreduziblen Elemente und die Linksmodularität dieser Ordnungen. Weiterhin definieren wir eine andere Beschriftung der kambrischen Halbverbände, mit deren Hilfe wir zeigen, dass diese Ordnungen beschränkt-homomorphe Bilder freier Verbände sind. Zudem präsentieren wir einen ersten Ansatz um beide Verallgemeinerungen zusammenzuführen. Das Ziel dieses Ansatzes ist es eine Verallgemeinerung der Tamariverbände zu definieren, die mit einer positiven Zahl m und einer Spiegelungsgruppe W parametrisiert ist, so dass man im Fall $m = 1$ die entsprechenden kambrischen Halbverbände erhält, und im Fall dass W die symmetrischen Gruppe ist die m-Tamariverbände erhält. Dafür definieren wir die sogenannte m-Bedeckungsordnung einer

gegebenen Ordnung. Diese Konstruktion funktioniert für die symmetrische Gruppe und die Diedergruppen, aber sie kann nicht direkt auf die anderen Spiegelungsgruppen übertragen werden.

Notation

$\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$ .... natural numbers, integers, real numbers, complex numbers

$\mathrm{Cat}(n), \mathrm{Cat}^{(m)}(n)$ .... (Fuß-)Catalan numbers

$\mathcal{T}_n, \mathcal{T}_n^{(m)}$ .... (m-)Tamari lattice of parameter n

A_{n-1} .... the symmetric group on $\{1, 2, \ldots, n\}$

$I_2(k)$ .... the dihedral group of order $2k$

W, S, T .... Coxeter group W, simple reflections, all reflections

ℓ_S, ℓ_T .... word length with respect to S or T

$\leq_S, \leq_T$ .... weak order, absolute order

$\mathcal{W}$ .... weak order semilattice of W

γ .... Coxeter element

$\gamma(w)$ .... γ-sorting word of w

C_γ .... γ-sortable elements

$\mathcal{C}_\gamma$ .... γ-Cambrian semilattice

$\pi_\downarrow^\gamma$ .... projection map from W to C_γ

$NC_W, NC_W^{(m)}$ .... (m-divisible) W-noncrossing partitions

$\mathcal{NC}_W, \mathcal{NC}_W^{(m)}$ .... poset of (m-divisible) W-noncrossing partitions

$\mu_\mathcal{P}$ .... Möbius function of $\mathcal{P}$

$\mathcal{P}^{\langle m \rangle}$ .... m-cover poset of $\mathcal{P}$

DM .... Dedekind-MacNeille completion

CHAPTER 0

Prologue

0.1. Fuß-Catalan Numbers

One of the most remarkable and most frequently occurring integer sequences in combinatorics is the sequence of *Catalan numbers*, see [**109**, A000108]. These numbers are defined by

$$\mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}, \tag{0.1}$$

and there exists a plethora of combinatorial interpretations of these numbers. Stanley has compiled a list of more than 200 combinatorial objects that are counted by these numbers, see [**110**]. Whenever we encounter a family of mathematical objects where the cardinality of its n-th member is given by $\mathrm{Cat}(n)$, then we refer to these as *Catalan objects*. Perhaps the oldest of these Catalan objects is the set of triangulations of a convex $(n+2)$-gon. In this thesis we encounter several other Catalan objects, such as Dyck paths of length $2n$, 312-avoiding permutations of $\{1,2,\ldots,n\}$, or noncrossing set partitions of $\{1,2,\ldots,n\}$.

Some years after Euler proposed that the number of triangulations of a convex $(n+2)$-gon is given by $\mathrm{Cat}(n)$, his secretary, by the name of Fusz, solved a more general problem: he showed that the number of ways to dissect a convex $(mn+2)$-gon into $(m+2)$-gons is given by

$$\mathrm{Cat}^{(m)}(n) = \frac{1}{mn+1}\binom{(m+1)n}{n}, \tag{0.2}$$

and these numbers are nowadays called the *Fuß-Catalan numbers*. Analogously to before we speak of a *Fuß-Catalan object* when we mean a family of mathematical objects indexed by two positive integers m and n such that the cardinality of their m,n-th member is given by $\mathrm{Cat}^{(m)}(n)$. In particular if $m=1$ these objects reduce to Catalan objects. Sometimes we call a Fuß-Catalan object a *Fuß-Catalan generalization* of some Catalan object. Besides the previously mentioned $(m+2)$-angulations of a convex $(mn+2)$-gon, the m-Dyck paths of length $(m+1)n$ or the m-divisible noncrossing set partitions of $\{1,2,\ldots,mn\}$ are well-known Fuß-Catalan objects. They are at the same time Fuß-Catalan generalizations of some Catalan objects mentioned in the previous paragraph.

The Fuß-Catalan numbers have a remarkable connection to the symmetric group[1] on $\{1,2,\ldots,n\}$, which we will denote by A_{n-1}. Let A_{n-1} act on $\mathbb{C}^n$ by permuting the standard basis vectors, i.e. for $\mathbf{v} = (v_1, v_2, \ldots, v_n)^\mathsf{T} \in \mathbb{C}^n$ and $\pi \in A_{n-1}$ we have $\pi \cdot \mathbf{v} = (v_{\pi(1)}, v_{\pi(2)}, \ldots, v_{\pi(n)})^\mathsf{T}$. We notice that A_{n-1} fixes the subspace $\{(v_1, v_2, \ldots, v_n)^\mathsf{T} \in \mathbb{C}^n \mid v_1 = v_2 = \cdots = v_n\}$ pointwise. Hence A_{n-1} acts essentially on the $(n-1)$-dimensional vector space

$$V = \big\{(v_1, v_2, \ldots, v_n)^\mathsf{T} \in \mathbb{C}^n \mid v_1 + v_2 + \cdots + v_n = 0\big\},$$

i.e. it fixes no point of V except the origin. Consider now the ring $\mathbb{C}[x_1, x_2, \ldots, x_n]$ of complex polynomials in n variables, and let A_{n-1} act on these polynomials by permuting the indices, i.e.

$$\pi \cdot f(x_1, x_2, \ldots, x_n) = f(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)})$$

for $\pi \in A_{n-1}$ and $f \in \mathbb{C}[x_1, x_2, \ldots, x_n]$. Let $\mathbb{C}[x_1, x_2, \ldots, x_n]^{A_{n-1}}$ denote the subring of $\mathbb{C}[x_1, x_2, \ldots, x_n]$ that consists of complex polynomials in n variables invariant under the action of A_{n-1}, i.e. those $f \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ with $\pi \cdot f = f$ for all $\pi \in A_{n-1}$. It is well known that this ring is for instance generated by the *elementary symmetric polynomials*

$$e_j(x_1, x_2, \ldots, x_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq n} x_{i_1} x_{i_2} \cdots x_{i_j},$$

for $j \in \{1,2,\ldots,n\}$. Moreover, if we recall that $\mathbb{C}[x_1, x_2, \ldots, x_n]$ is isomorphic to the symmetric algebra $\mathrm{Sym}(\mathbb{C}^n)$, then it follows that $\mathrm{Sym}(\mathbb{C}^n)^{A_{n-1}} \cong \mathbb{C}[e_1, e_2, \ldots, e_n]$. If we restrict our attention to the dual of the essential space of A_{n-1}, then we obtain $\mathrm{Sym}(V^*)^{A_{n-1}} \cong \mathbb{C}[e_2, e_3, \ldots, e_n]$, since e_1 vanishes on V. Let us write $d_j = j+1$ for $j \in \{1,2,\ldots,n-1\}$. We notice that d_j is precisely the degree of e_{j+1}. It is well known that these degrees are invariants of A_{n-1}, i.e. any minimal set of homogeneous polynomials generating $\mathrm{Sym}(V^*)^{A_{n-1}}$ (like complete symmetric functions or Schur functions) has these numbers as set of degrees, see for instance [**63**, Proposition 3.7]. Thus it is justified to call these numbers the *degrees of the symmetric group*. With their help we can factorize $\mathrm{Cat}^{(m)}(n)$ as follows:

$$\begin{aligned}
\mathrm{Cat}^{(m)}(n) &= \frac{1}{mn+1}\binom{(m+1)n}{n} \\
&= \frac{\big((m+1)n\big)\big((m+1)n-1\big)\cdots(mn+2)}{n(n-1)\cdots 2} \\
&= \frac{mn+n}{n}\cdot\frac{mn+(n-1)}{n-1}\cdot\ldots\cdot\frac{mn+2}{2} \\
&= \frac{mn+2}{2}\cdot\frac{mn+3}{3}\cdot\ldots\cdot\frac{mn+n}{n} \\
&= \prod_{i=1}^{n-1}\frac{md_n+d_i}{d_i}.
\end{aligned}$$

The existence of such invariants is not an exclusive property of the symmetric group, but it holds in every finite irreducible complex reflection group. It is a well-known theorem by Shephard and Todd that the ring of complex polynomials invariant under the action of a finite group $G \subseteq GL_n(\mathbb{C})$ is generated by polynomials if and only if G is an irreducible complex reflection group, see [**104**, Proposition 5.1]. While Shephard and Todd proved this theorem in a case-by-case fashion, the first uniform proof of this result was given by Chevalley in [**39**]. Again the set of degrees of these generating polynomials does not depend on the actual

[1] Throughout this thesis we use the Coxeter notation A_{n-1} for the symmetric group on $\{1,2,\ldots,n\}$. The shift in the indices comes from the fact that as a reflection group it acts essentially on an $(n-1)$-dimensional space.

choice of these generators. Hence we can associate a set of *degrees* with every finite complex reflection group W, and we can define the *W-Fuß-Catalan numbers* by

$$\mathrm{Cat}^{(m)}(W) = \prod_{i=1}^{n} \frac{md_n + d_i}{d_i}, \tag{0.3}$$

where the numbers $d_1, d_2, \ldots, d_n$ denote the degrees of W in nondecreasing order. In the case $m = 1$ we call those numbers the *W-Catalan numbers*, and we write $\mathrm{Cat}(W)$.

For $m = 1$ the formula in (0.3) was first written down in [**44**, Theorem 3], but without the connection to the Catalan numbers. This connection was first observed for W a Coxeter group by Reiner in [**98**, Remark 2] for $m = 1$, and by Athanasiadis in [**4**, Corollary 1.3] for $m > 1$. Bessis finally proposed this formula for all well-generated complex reflection groups in [**13**, Section 13]. We define *Coxeter-(Fuß-)Catalan objects* and *Coxeter-(Fuß-)Catalan generalizations* analogously to before. (Here the term "Coxeter" refers to the fact that the well-generated complex reflection groups are geometric generalizations of finite Coxeter groups.)

The observation that the Fuß-Catalan numbers can be generalized to all finite well-generated complex reflection groups opened up a whole new field of research, which can best be named by *Fuß-Catalan combinatorics*. The purpose of this field of research is to find new Coxeter-(Fuß-)Catalan generalizations of known Catalan objects, and to use these generalizations to exhibit new, or to explain old connections between such objects. This research was initiated in the late 1990s when noncrossing set partitions were generalized to well-generated complex reflection groups, see [**1**, **7**, **32**, **33**, **98**]. This generalization has had quite some impact on many different areas of mathematics, such as group theory [**19**], topology [**12**, **15**, **32**], free probability [**18**], representation theory of quivers [**64**], or cluster algebras [**94**]. See Section 4.1 for some further explanation. Soon after, more classical Catalan objects were generalized analogously, such as nonnesting set partitions [**98**, Remark 2] or [**5**], triangulations [**53**, **54**], or 312-avoiding permutations [**94**]. It turned out that these Coxeter-Catalan generalizations share many combinatorial and enumerative properties, and we refer to Armstrong's excellent exposition on this topic in [**1**, Chapter 5] for more details.

0.2. Fuß-Catalan Posets

In this thesis we investigate two different families of Fuß-Catalan objects equipped with a partial order: on the one hand generalizations of the Tamari lattice and on the other hand the lattices of noncrossing partitions.

0.2.1. Generalized Tamari Posets. The Tamari lattice of parameter n, denoted by $\mathcal{T}_n$, was introduced by Tamari as a partial order on so-called bracketings of strings where the cover relations are given by applications of a semiassociative law [**118**]. It was shown in [**119**] that the cardinality of $\mathcal{T}_n$ is given by $\mathrm{Cat}(n)$, and in view of the many known Catalan objects it is not surprising that there exist many realizations of $\mathcal{T}_n$ as a poset (short for "partially ordered set") on such Catalan objects. We are mainly interested in two of these realizations: (i) via Dyck paths of length $2n$ equipped with the rotation order and (ii) via 312-avoiding permutations of $\{1, 2, \ldots, n\}$ equipped with the weak order. While the realization in terms of Dyck paths is quite straightforward from Tamari's original definition, the realization in terms of 312-avoiding permutations is less obvious, see [**26**, Theorem 9.6(i) and (ii)]. Our interest in these realizations comes from the fact that each of them is the starting point of a generalization of $\mathcal{T}_n$: (i) Bergeron and Préville-Ratelle defined in [**11**] the *m-Tamari lattice $\mathcal{T}_n^{(m)}$* as a poset on m-Dyck paths equipped with rotation order, and (ii) Reading defined in [**95**] the *Cambrian lattices $\mathcal{C}_\gamma$* as posets on γ-sortable elements for some Coxeter element γ in a Coxeter group W. (There is a particular choice of W and γ such that one obtains precisely the 312-avoiding permutations.) A more detailed explanation of these generalizations can be found in Sections 2.1 and 3.1, respectively.

For now it is sufficient to state that $\mathcal{T}_n^{(m)}$ can be seen as a Fuß-Catalan generalization of $\mathcal{T}_n$, and $\mathcal{C}_\gamma$ can be seen as a Coxeter-Catalan generalization of $\mathcal{T}_n$. It is an immediate question whether these generalizations preserve the properties of $\mathcal{T}_n$, and, if so, to which extent. One of the main contributions of this thesis is an affirmative answer to these questions from a topological and from a structural viewpoint. When we speak of the topology of a poset, we mean the topology of a certain simplicial complex of this poset, the so-called *order complex*, and it follows from Rota's work in [**101**] that the *Möbius function* of a poset is closely related to its topology.

In the case of the Tamari lattices it was observed by Pallo in [**89**, Section 5] that the Möbius function of $\mathcal{T}_n$ takes only values in $\{-1,0,1\}$. Later Björner and Wachs proved that $\mathcal{T}_n$ is EL-shellable, which implies that its order complex is homotopy equivalent to a sphere. We prove that the same is true for $\mathcal{T}_n^{(m)}$ and for $\mathcal{C}_\gamma$, see Theorem 2.3.1 on page 33 and Theorem 3.4.1 on page 78, and we characterize the intervals of these posets according to the value the Möbius function takes on them. These results are obtained by defining a certain edge-labeling of these posets, which then allows for the application of a whole combinatorial framework developed mainly by Björner and Wachs in [**21**,**24**–**26**]. We remark that for some special cases these results were already known before, and we refer the reader to Sections 2.3 and 3.4, respectively, for a more detailed exposition.

On a structural level it was observed by Urquhart in [**125**] that $\mathcal{T}_n$ is a so-called *bounded-homomorphic image of a free lattice*. While this result can be easily transferred to $\mathcal{T}_n^{(m)}$, it is not so obvious for $\mathcal{C}_\gamma$. Again by using a certain kind of edge-labeling we prove that the same is true for $\mathcal{C}_\gamma$, see Theorem 3.5.1 on page 88. Moreover, we show that the irreducibles of $\mathcal{C}_\gamma$ satisfy the same extremality condition as the irreducibles of $\mathcal{T}_n$, see Theorem 3.4.14 on page 84.

We have remarked earlier that $\mathcal{T}_n^{(m)}$ and $\mathcal{C}_\gamma$ are two different, in a sense orthogonal, generalizations of $\mathcal{T}_n$. It is an intriguing question, whether these two generalizations can be combined. More precisely, this question can be phrased as follows: what would a Coxeter-Fuß-Catalan object look like that simultaneously generalizes $\mathcal{T}_n^{(m)}$ and $\mathcal{C}_\gamma$? This object would have to be parametrized by a Coxeter group W, a Coxeter element γ, and a positive integer m such that for $m=1$ we obtain $\mathcal{C}_\gamma$ and for $W=A_{n-1}$ we obtain $\mathcal{T}_n^{(m)}$. We present a first partial solution to this question that works nicely for the dihedral groups, see Section 2.4.4. However, our construction cannot be generalized directly to other Coxeter groups, so this question remains open.

0.2.2. Lattices of Noncrossing Partitions. The lattices of noncrossing partitions have a huge impact on many fields of mathematics, as briefly announced at the end of Section 0.1. For a more detailed introduction on the history of these lattices, we refer to Section 4.1 or to the surveys [**77**,**106**]. For now, let W be a finite well-generated complex reflection group, let $\gamma\in W$ be a Coxeter element, and let $\mathcal{NC}_W(\gamma)$ denote the corresponding lattice of noncrossing partitions. If W is a Coxeter group, there is an interesting connection between $\mathcal{NC}_W(\gamma)$ and the elements of $\mathcal{C}_\gamma$: namely if we equip the elements of $\mathcal{C}_\gamma$, the *γ-sortable elements of W*, with a certain partial order, the *shard intersection order*, then we obtain precisely $\mathcal{NC}_W(\gamma)$, see [**96**, Theorem 8.5]. However, we do not elaborate on this connection here.

Instead we are interested in Armstrong's generalization of $\mathcal{NC}_W$ to a poset of *m-divisible W-noncrossing partitions*, denoted by $\mathcal{NC}_W^{(m)}$, and again we investigate these posets from a topological point of view. It has been shown uniformly for W a Coxeter group that $\mathcal{NC}_W^{(m)}$ is EL-shellable, see [1,6], and what the possible values of the Möbius function of $\mathcal{NC}_W^{(m)}$ are, see [2]. However, in general the reasoning in these proofs cannot be transferred to well-generated complex reflection groups. We close this gap by providing a proof that $\mathcal{NC}_W^{(m)}$ is EL-shellable for all well-generated complex reflection groups by means of a case-by-case analysis, see Theorem 4.4.1 on page 109. A uniform solution to this question remains open, but we suggest a possible uniform approach in Section 4.4.5.

0.3. On the Organization of this Thesis

This thesis consists mainly of four parts. The first part, Chapter 1, gives all the background information on posets, poset topology, and Coxeter groups that is necessary for the understanding of this thesis. Readers familiar with these topics can skip this chapter, and return back to it whenever some explanation is in order. The index at the end of this thesis might also help with quickly finding the places of definition of the concepts used here.

The remaining three parts, Chapters 2–4, can be seen as independent from each other, so that it does not really matter in which order they are read. There is perhaps a slight dependence between Chapter 2 and Chapter 3, however not to an extent that requires having read the one chapter first in order to understand the other.

More precisely, in Chapter 2 we formally define the m-Tamari lattices $\mathcal{T}_n^{(m)}$, we investigate their topology, and we present a generalization of $\mathcal{T}_n^{(m)}$ to the dihedral groups. These two aspects are described separately in our articles [83] and [67], respectively, where also some additional results are provided. In Chapter 3 we formally define the Cambrian semilattices, we investigate their topology, and we prove several structural properties. Again these results are described separately in [66] and [85], respectively, and some additional results are provided there. Finally, in Chapter 4, we formally define complex reflection groups as well as the lattices of noncrossing partitions, and we investigate their topology. The exposition of this chapter follows [82]. Each of these three chapters has a separate introductory section in which we give a brief historical outline of the central objects and references to related work.

In the epilogue, Chapter 5, we outline potential future research with which we intend to continue the work presented in this thesis.

CHAPTER 1

Basic Notions

1.1. Posets and Lattices

In this section we briefly recall the necessary definitions concerning partially ordered sets that we need in this thesis. A good textbook on this subject matter is for instance [**42**]. For an introduction to partially ordered sets with a more combinatorial emphasis we refer to [**112**, Chapter 3]. Even though some of the partially ordered sets occurring in this thesis are infinite per se, we will always reduce our investigation to a finite subposet of those. Thus we assume everything to be finite, unless explicitly stated.

1.1.1. Partially Ordered Sets. We begin with the very basic notions. Given a set P, a *partial order on P* is a binary relation $\leq$ on P having the following three properties:

(i) if $p \in P$, then $p \leq p$; (Reflexivity)
(ii) if $p, q \in P$ as well as $p \leq q$ and $q \leq p$, then $p = q$; (Antisymmetry)
(iii) if $p, q, r \in P$ as well as $p \leq q$ and $q \leq r$, then $p \leq r$. (Transitivity)

Then we call the pair $(P, \leq)$ a *partially ordered set*, or *poset* for short, and we usually denote it by $\mathcal{P}$. The relation $\leq$ is said to be a *total order on P* if it additionally has the following property:

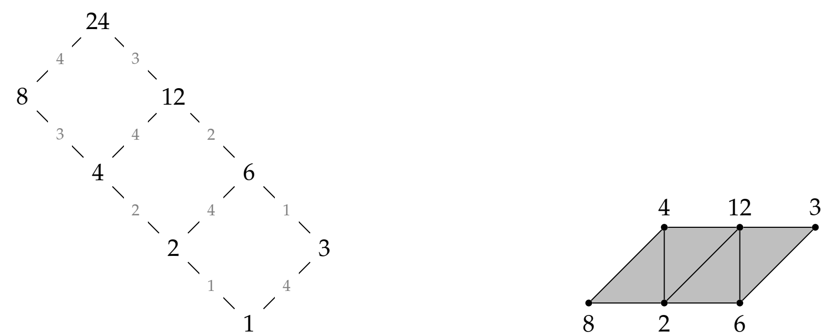
(iv) if $p, q \in P$, then $p \leq q$ or $q \leq p$, (Totality)

and we call $\mathcal{P}$ a *totally ordered set*. We use the obvious conventions that $p \geq q$ simply means $q \leq p$, and $p < q$ means $p \leq q$ and $p \neq q$. If neither $p \leq q$ nor $q \leq p$ holds, then we write $p \parallel q$ instead.

For a subset $P' \subseteq P$ the restriction of $\leq$ to P' produces an induced partial order $\leq'$, and we call the pair $(P', \leq')$ a *subposet of $\mathcal{P}$*. A frequently recurring poset feature is *duality*: if $\leq$ is a partial order on P, then we define the *dual order of $\leq$* by $p \leq^d q$ if and only if $q \leq p$, and we call the poset $(P, \leq^d)$ the *dual poset of $(P, \leq)$*.

An element $p \in P$ is *minimal in $\mathcal{P}$* if there is no element $q \in P$ with $q < p$. Dually, p is *maximal in $\mathcal{P}$* if there is no element $q \in P$ with $p < q$. If there is only one minimal element in $\mathcal{P}$, then this is the *least element of $\mathcal{P}$*, and dually, if there is only one maximal element in $\mathcal{P}$, then this is the *greatest element of $\mathcal{P}$*. If $\mathcal{P}$ has both a least and a greatest element, usually denoted by $\hat{0}$ and $\hat{1}$, then we call $\mathcal{P}$ *bounded*. The subposet of $\mathcal{P}$ induced by $\overline{P} = P \setminus \{\hat{0}, \hat{1}\}$ is called the *proper part of $\mathcal{P}$*, and it is usually denoted by $\overline{\mathcal{P}}$.

Now let $p, q \in P$ with $p \leq q$. The set $[p, q] = \{r \in P \mid p \leq r \leq q\}$ is called a *closed interval of $\mathcal{P}$*, and the set $(p, q) = \{r \in P \mid p < r < q\}$ is called an *open interval of $\mathcal{P}$*. The half-open



(a) The poset $\big(D(24), |\big)$.

(b) The order complex of $\big(D(24), |\big)$.

FIGURE 1. A poset and its order complex.

intervals $(p, q]$ and $[p, q)$ are defined analogously. If the cardinality of the closed interval $[p, q]$ is two, then we say that *q covers p*, or equivalently that *p is covered by q*, and we write $p \lessdot q$. The set of *cover relations of $\mathcal{P}$* is $\mathcal{E}(\mathcal{P}) = \big\{(p, q) \in P \times P \mid p \lessdot q\big\}$. It is common practice to graphically represent $\mathcal{P}$ via its *Hasse diagram*. This is a graph with vertex set P and edge set $\mathcal{E}(\mathcal{P})$, with the convention that if $p < q$, then p is drawn strictly below q. Clearly the partial order $\leq$ can be recovered from $\mathcal{E}(\mathcal{P})$ by taking the reflexive and transitive closure. If $\mathcal{P}$ has a least element $\hat{0}$, then we call an element $p \in P$ with $\hat{0} \lessdot p$ an *atom of $\mathcal{P}$*. Dually, if $\mathcal{P}$ has a greatest element $\hat{1}$, then we call an element $p \in P$ with $p \lessdot \hat{1}$ a *coatom of $\mathcal{P}$*.

A totally ordered subset $C \subseteq P$ is called a *chain of $\mathcal{P}$*. This implies that we can uniquely write $C = \{p_0, p_1, \ldots, p_s\}$ where $p_i < p_j$ if and only if $i < j$ for all $i, j \in \{0, 1, \ldots, s\}$. We usually write such a chain as $C : p_0 < p_1 < \cdots < p_s$. Moreover, a chain is *saturated* if it can be written as $C : p_0 \lessdot p_1 \lessdot \cdots \lessdot p_s$, and a chain is *maximal in the interval $[p, q]$* if it is saturated and $p_0 = p$ and $p_s = q$. The *length of a chain* is its cardinality minus one, and the *length of $\mathcal{P}$* is the maximal length of a maximal chain in $\mathcal{P}$. We usually denote the length of $\mathcal{P}$ by $\ell(\mathcal{P})$. Finally we say that a bounded poset is *graded* if all maximal chains have the same length. Graded posets admit a rank function rk, which is defined by $\mathrm{rk}(p) = \ell([\hat{0}, p])$.

EXAMPLE 1.1.1

Let n be a positive integer, let $D(n)$ denote the set of divisors of n, and consider this set ordered by divisibility. For instance, if $n = 24$, then we have $D(24) = \{1, 2, 3, 4, 6, 8, 12, 24\}$. It is immediately clear that the poset $\big(D(6), |\big)$ is a subposet of $\big(D(24), |\big)$. Moreover, we have

$$\mathcal{E}\big(\mathcal{D}_{24}\big) = \big\{(1,2), (1,3), (2,4), (2,6), (3,6), (4,12), (4,8), (6,12), (8,24), (12,24)\big\},$$

and the Hasse diagram of $\big(D(24), |\big)$ is shown in Figure 1(a). We notice that this poset is graded, and its rank function is given by

$$\mathrm{rk}(1) = 0, \quad \mathrm{rk}(2) = \mathrm{rk}(3) = 1, \quad \mathrm{rk}(4) = \mathrm{rk}(6) = 2, \quad \mathrm{rk}(8) = \mathrm{rk}(12) = 3, \quad \mathrm{rk}(24) = 4.$$

The set $\{1, 2, 4, 8, 24\}$ is a maximal chain in this poset.

1.1.2. Poset Topology. One of the main contributions of this thesis is the investigation of the topology of certain families of posets, i.e. the investigation of a topological space associated with each of these posets. The topological space in question is given by a simplicial complex, the so-called order complex, which is defined via the chains of the given poset. We will now briefly define the necessary concepts. For a more detailed exposition of this topic, including any undefined concepts and historical remarks, we refer to [**22**] or [**127**].

For a finite set M we say that an *(abstract) simplicial complex on M* is a nonempty collection Δ of subsets of M such that $\{m\} \in \Delta$ for all $m \in M$, and if $F \in \Delta$ and $F' \subseteq F$, then $F' \in \Delta$. The elements of M are called the *vertices of Δ*, the elements of Δ are called the *faces of Δ*, and the maximal faces (with respect to inclusion) are called the *facets of Δ*. The *dimension of a face* is its cardinality minus one, and the *dimension of Δ* is the maximum of the dimensions of its facets. In both cases we use the abbreviation dim. (Note that the empty set is always a face of dimension -1.) If all facets of Δ have the same dimension, then we call Δ *pure*.

With each abstract simplicial complex Δ we can associate a topological space as follows. First we say that a *d-simplex in $\mathbb{R}^n$* is the convex hull of $d+1$ affinely independent vectors in $\mathbb{R}^n$, and the *geometric realization of Δ* is then the union of the simplices defined by the faces of Δ. More precisely, if F is a face of Δ with dimension d, then we associate a d-simplex with this face, and whenever two faces of Δ have a common subface, then the corresponding simplices have a common subsimplex as well. Moreover, we say that a simplicial complex is *contractible* if its geometric realization is homotopy equivalent to a point, i.e. it can be deformed continuously into a point. It is called *spherical* if it is homotopy equivalent to a sphere.

The simplicial complexes that we will consider in this thesis are constructed from posets in the following way: given a poset $\mathcal{P} = (P, \leq)$, we say that the *order complex of $\mathcal{P}$*, denoted by $\Delta(\mathcal{P})$, is the simplicial complex whose vertex set is P, and whose faces are the chains of $\mathcal{P}$. It is immediate that the dimension of $\Delta(\mathcal{P})$ is precisely $\ell(\mathcal{P})$, and that $\Delta(\mathcal{P})$ is pure if and only if $\mathcal{P}$ is graded. The topological space associated with a $\mathcal{P}$ (i.e. the topological space that we are interested in) is then the geometric realization of the order complex of $\mathcal{P}$.

Example 1.1.2

The simplicial complex shown in Figure 1(b) is precisely the order complex of the proper part of the poset $\big(D(24), |\big)$ from Figure 1(a), and likewise the simplicial complex shown in Figure 2(b) is precisely the order complex of the proper part of the poset shown in Figure 2(a).

Next we will describe the main properties of simplicial complexes in which we are interested in this thesis. If we denote by $f_i(\Delta)$ the number of faces of Δ having dimension i, then we can define the *reduced Euler characteristic of Δ* by

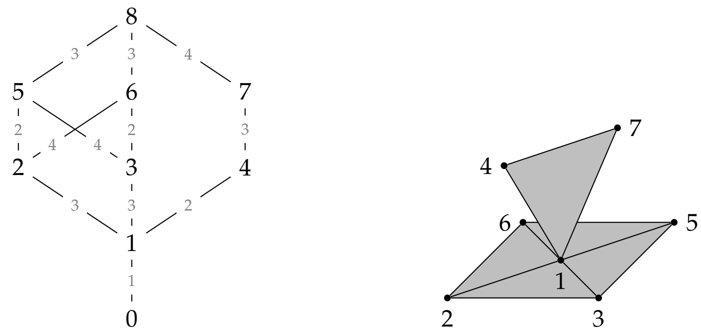
$$\tilde{\chi}(\Delta) = \sum_{i=-1}^{\dim(\Delta)} (-1)^i f_i(\Delta).$$

The reduced Euler characteristic is an important topological invariant, since it has a deep connection to the homology of the geometric realization of Δ. Recall that the *i-th reduced Betti number $\tilde{\beta}_i(\Delta)$* is defined as the rank of the i-th homology group of the geometric realization of Δ. The previously mentioned connection is given by the following well-known formula.

Theorem 1.1.3 ([**127**, Theorem 1.2.8])

For any simplicial complex Δ we have

$$\tilde{\chi}(\Delta) = \sum_{i=-1}^{\dim(\Delta)} (-1)^i \tilde{\beta}_i(\Delta).$$



(a) A nonshellable poset.

(b) The order complex of the poset in Figure 2(a).

Figure 2. A nonshellable poset and its order complex.

Example 1.1.4

Consider the set $M = \{2,3,4,6,8,12\}$, and define an abstract simplicial complex Δ via the facets

$$F_1 = \{2,4,8\}, \quad F_2 = \{2,4,12\}, \quad F_3 = \{2,6,12\}, \quad F_4 = \{3,6,12\}.$$

The corresponding geometric realization of Δ is shown in Figure 1(b). The shaded regions indicate two-dimensional simplices. The face numbers of this simplicial complex are

$$f_{-1}(\Delta) = 1, \quad f_0(\Delta) = 6, \quad f_1(\Delta) = 9, \quad f_2(\Delta) = 4,$$

and we obtain $\tilde{\chi}(\Delta) = -1+6-9+4 = 0$. Moreover, the geometric realization of Δ is a convex subset of $\mathbb{R}^2$, and thus contractible, which implies that $\tilde{\beta}_j = 0$ for all j.

Let us now introduce a class of simplicial complexes with a particularly beautiful structure. For a face $F \in \Delta$ denote by $\overline{F}$ the set of subfaces of F, which is a simplicial complex in its own right. We say that Δ is *shellable* if its facets can be arranged in a linear order $F_1 \prec F_2 \prec \cdots \prec F_s$ such that $\left(\bigcup_{i=1}^{k-1} \overline{F}_i\right) \cap \overline{F}_k$ is a pure simplicial complex of dimension $\dim(F_k) - 1$ for all $k \in \{2,3,\ldots,s\}$. In this case we call this particular linear order a *shelling order* of the facets of Δ. Moreover, we say that a poset is *shellable* if its order complex is shellable. Shellable simplicial complexes have several nice topological properties, one of them is stated in the next well-known theorem.

Theorem 1.1.5 ([25, Theorem 4.1])

Let Δ be a shellable complex of dimension d. Then, Δ has the homotopy type of a wedge of $\tilde{\beta}_j(\Delta)$-many j-dimensional spheres for all $j \in \{0,1,\ldots,d\}$.

Let us recall a helpful result on the ordering of the dimensions of the facets in a shelling order.

Lemma 1.1.6 ([25, Lemma 2.2])

If Δ is a shellable simplicial complex and $F_1 \prec F_2 \prec \cdots \prec F_s$ is a shelling order of its facets, then $\dim(F_1) = \dim(\Delta)$.

In particular, for every shellable complex we can find a shelling order in which the facets are ordered weakly decreasingly with respect to their dimensions.

EXAMPLE 1.1.7

The ordering $\{2,4,8\} \prec \{2,4,12\} \prec \{2,6,12\} \prec \{3,6,12\}$ of the facets of the simplicial complex Δ from Example 1.1.4 is indeed a shelling order of Δ, since two consecutive facets intersect in exactly two points, which then form a one-dimensional pure simplicial complex, and earlier facets do not contribute additional vertices to this intersection.

Now consider the simplicial complex Δ' on the set $M = \{1,2,3,4,5,6,7\}$ whose facets are

$$F_1 = \{1,2,3\}, \quad F_2 = \{1,2,6\}, \quad F_3 = \{1,3,5\}, \quad F_4 = \{1,5,6\}, \quad F_5 = \{1,4,7\}.$$

Its geometric realization is shown in Figure 2(b). The intersection of the facet F_5 with each of the other facets is exactly the point $\{1\}$ and thus zero-dimensional. Since $\dim(F_5) = 2$ it follows that there cannot exist a shelling order of the facets of Δ', which is thus nonshellable.

1.1.3. Möbius Function. The main connection between the combinatorial and the topological part of this thesis is the well-known Möbius function, and this connection was first worked out in detail by ROTA in his seminal paper [**101**]. Again we start with the basic definitions.

DEFINITION 1.1.8

Let $\mathcal{P} = (P, \leq)$ be a poset. The *Möbius function of $\mathcal{P}$* is the function $\mu_{\mathcal{P}} : P \times P \to \mathbb{N}$ that is recursively defined by:

$$\mu_{\mathcal{P}}(p,q) = \begin{cases} 1, & \text{if } p = q, \\ -\sum\limits_{p<r\leq q} \mu(r,q), & \text{if } p < q, \\ 0, & \text{otherwise.} \end{cases}$$

REMARK 1.1.9

The Möbius function of a poset is closely related to the classical Möbius function from number theory, which is defined as the function $\mu : \mathbb{N} \to \{-1,0,1\}$ with

$$\mu(n) = \begin{cases} (-1)^k, & \text{if } n \text{ has } k \text{ distinct prime factors,} \\ 0, & \text{otherwise.} \end{cases}$$

In particular, if we write $\mathcal{D}_n = \big(D(n), |\big)$, then for $d \in \mathbb{N}$ with $d \mid n$ we have $\mu_{\mathcal{D}_n}(d,n) = \mu(\frac{n}{d})$.

In this thesis we will mainly consider bounded posets, and for our purposes the most important value of the Möbius function will be the so-called *Möbius invariant of $\mathcal{P}$*, namely $\mu(\mathcal{P}) = \mu_{\mathcal{P}}(\hat{0}, \hat{1})$.

EXAMPLE 1.1.10

The Möbius function of the poset $\mathcal{D}_{24}$ from Figure 1(a) can be described best in terms of a matrix where the rows and columns are indexed by the elements of $D(24)$ in increasing order:

$$\mu_{\mathcal{D}_{24}} = \begin{pmatrix} 1 & -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus the entry -1 in the second row and fifth column means that $\mu_{\mathcal{D}_{24}}(2,6) = -1$. Since $\frac{6}{2} = 3$ is already prime it follows that $\mu(3) = -1$. Moreover, we have $\mu(\mathcal{D}_{24}) = 0$, and since $24 = 2^3 \cdot 3$ it follows that $\mu(24) = 0$.

An important combinatorial way to compute the Möbius invariant of finite bounded posets is given in the next result, which first appeared in [61]. For reasons of availability we give a different reference here.

Proposition 1.1.11 ([112, Proposition 3.8.5])

Let $\mathcal{P}$ be a finite bounded poset, and let c_i denote the number of maximal chains in $\mathcal{P}$ having length i. Then,

$$\mu(\mathcal{P}) = \sum_{i=0}^{\ell(\mathcal{P})} (-1)^i c_i.$$

Thus by definition of the order complex, we immediately obtain the following equivalent formulation of Proposition 1.1.11.

Proposition 1.1.12 ([112, Proposition 3.8.6])

Let $\mathcal{P}$ be a finite bounded poset. Then, $\mu(\mathcal{P}) = \tilde{\chi}\big(\Delta(\overline{\mathcal{P}})\big)$.

1.1.4. Edge-Labelings. The main tools used in this thesis are certain poset edge-labelings. First we define edge-labelings in general, and then we introduce the two types of edge-labelings that we use later on.

Let $\mathcal{P} = (P, \leq)$ be a bounded poset, and let $(\Lambda, \leq_\Lambda)$ be another poset. A map $\lambda : \mathcal{E}(\mathcal{P}) \to \Lambda$ is called an *edge-labeling of $\mathcal{P}$*. If $C : p_0 \lessdot p_1 \lessdot \cdots \lessdot p_s$ is a saturated chain of $\mathcal{P}$, then we abbreviate the sequence $\big(\lambda(p_0,p_1), \lambda(p_1,p_2), \ldots, \lambda(p_{s-1},p_s)\big)$ by $\lambda(C)$. We say that a chain C of $\mathcal{P}$ is *rising (with respect to λ)* if the sequence $\lambda(C)$ is strictly increasing with respect to $\leq_\Lambda$. Analogously, we say that C is *falling (with respect to λ)* if $\lambda(C)$ is weakly decreasing with respect to $\leq_\Lambda$. Let $\Lambda^\star$ denote the set of tuples of elements of Λ having arbitrary length. (Sometimes this is referred to as the set of words on the alphabet Λ.) We define the *lexicographic order on $\Lambda^\star$* by $(u_1, u_2, \ldots, u_s) \leq_{\mathrm{lex}} (v_1, v_2, \ldots, v_t)$ if and only if either

$$\begin{aligned} &u_i = v_i, && \text{for } i \in \{1,2,\ldots,s\} \text{ and } s \leq t, \quad \text{or} \\ &u_i <_\Lambda v_i, && \text{for the least } i \text{ such that } u_i \neq v_i. \end{aligned}$$

It is easy to check that $\leq_{\mathrm{lex}}$ is a total order.

DEFINITION 1.1.13

Let $\mathcal{P}$ be a bounded poset. An edge-labeling λ of $\mathcal{P}$ is called an *ER-labeling* (which is short for "edge rising labeling") if in every closed interval of $\mathcal{P}$ there exists a unique rising maximal chain. An ER-labeling is called an *EL-labeling* (which is short for "edge lexicographic labeling") if in every closed interval the unique rising chain is lexicographically first among all maximal chains in this interval. If $\mathcal{P}$ admits an EL-labeling, then $\mathcal{P}$ is called *EL-shellable*.

The notion of EL-labelings was introduced by BJÖRNER in [**21**] in order to prove a conjecture by STANLEY who asked whether every admissible lattice is Cohen-Macaulay. Admissible lattices are a certain class of graded lattices (see Section 1.1.5) that have an ER-labeling, and they were first considered in [**111**]. The power of ER-labelings comes from the following result, which was first stated in the graded lattice case by STANLEY, see [**111**, Corollary 3.3], and later generalized to the non-graded poset case by BJÖRNER and WACHS, see [**25**, Proposition 5.7].

PROPOSITION 1.1.14 ([**25**, Proposition 5.7])

Let $\mathcal{P}$ be a bounded poset admitting an ER-labeling, and let f_o denote the number of falling maximal chains from $\hat{0}$ to $\hat{1}$ with odd length, and let f_e denote the number of such chains with even length. Then, $\mu(\mathcal{P}) = f_e - f_o$.

It is clear from the definition that the restriction of an ER-labeling to some interval of $\mathcal{P}$ yields an ER-labeling of this interval, and hence we can compute every single value of $\mu_{\mathcal{P}}$ by counting falling chains between the corresponding poset elements. Moreover, the power of EL-labelings lies in the fact that they produce a shelling order of the order complex of $\mathcal{P}$ in the following way.

THEOREM 1.1.15 ([**25**, Theorem 5.8])

Let $\mathcal{P}$ be a bounded EL-shellable poset. The order complex $\Delta(\overline{\mathcal{P}})$ is shellable, and every linear extension of the lexicographic order on the maximal chains of $\mathcal{P}$ induces a shelling order on $\Delta(\overline{\mathcal{P}})$.

With the help of EL-labelings we can strengthen Theorem 1.1.5 as follows:

THEOREM 1.1.16 ([**25**, Theorem 5.9])

Let $\mathcal{P}$ be an EL-shellable poset, and let λ be some EL-labeling of $\mathcal{P}$. Then, $\tilde{\beta}_j\big(\Delta(\overline{\mathcal{P}})\big)$ equals the number of falling maximal chains in $\mathcal{P}$ having length $j+2$.

EXAMPLE 1.1.17

In Example 1.1.2 we have seen that the simplicial complex Δ from Figure 1(b) is shellable, and we observed in Example 1.1.17 that Δ is the order complex of the proper part of the poset $\mathcal{D}_{24}$ from Figure 1(a). It is indeed easy to verify that the labeling given there is an EL-labeling.

On the other hand, the simplicial complex Δ' from Figure 2(b) is nonshellable, and it is the order complex of the proper part of the poset $\mathcal{P}'$ shown in Figure 2(a). Thus Theorem 1.1.15 implies that $\mathcal{P}'$ cannot be EL-shellable, and we can quickly verify that the labeling given there is an ER-labeling but no EL-labeling, since the unique rising chain in the interval $[1,6]$ is not lexicographically first. On the other hand, since there needs to be a rising chain in the intervals $[1,7]$ and $[4,8]$, it follows that the chain $1 \lessdot 4 \lessdot 7 \lessdot 8$ must be rising, but we cannot modify the other labels such that we obtain an EL-labeling.

Now we introduce another poset edge-labeling that we use in Section 3.5.1. We call an interval $[p,q]$ of $\mathcal{P}$ a *2-facet* if there are exactly two maximal chains in $[p,q]$, and they intersect only in p and q. This implies in particular that (in $[p,q]$) the element p is covered by exactly two elements, say p_1 and p_2, and (in $[p,q]$) the element q covers exactly two elements, say q_1 and q_2. We can assume that these elements are labeled in such a way that $p_1 \leq q_1$ and $p_2 \leq q_2$.

Definition 1.1.18

Let $\mathcal{P}$ be a bounded poset. An edge-labeling λ of $\mathcal{P}$ is called a *2-facet labeling* if it satisfies $\lambda(p,p_1) = \lambda(q_2,q)$ and $\lambda(p,p_2) = \lambda(q_1,q)$ for every 2-facet $[p,q]$.

Example 1.1.19

The poset $\mathcal{D}_{24}$ from Figure 1(a) has exactly three 2-facets, namely the intervals $[1,6],[2,12]$ and $[4,24]$. Moreover, the edge-labeling given there is in fact both an EL-labeling and a 2-facet labeling.

1.1.5. Lattices. Most of the posets which we consider in this thesis come equipped with an additional property which we will describe next. Let $\mathcal{P} = (P,\leq)$ be a poset, and let $p,q \in P$. If the set $\{r \in P \mid p,q \leq r\}$ (which is a poset in its own right) has a least element, then we call this element the *join of p and q*, and we denote it by $p \vee q$. Dually, if the set $\{r \in P \mid r \leq p,q\}$ has a greatest element, then we call this element the *meet of p and q*, and we denote it by $p \wedge q$. A *join-semilattice* is a poset in which the join of any two elements exists, and dually, a *meet-semilattice* is a poset in which the meet of any two elements exists. A *lattice* is a poset that is both a join- and a meet-semilattice. A subposet $\mathcal{P}'$ of a lattice $\mathcal{P}$ is a *sublattice of $\mathcal{P}$* if $\mathcal{P}'$ is a lattice in its own right and if joins and meets in $\mathcal{P}'$ agree with joins and meets in $\mathcal{P}$. Subsemilattices are defined analogously.

Example 1.1.20

The poset in Figure 1(a) is in fact a lattice, while the poset in Figure 2(a) is not, since for instance the elements 2 and 3 do not have a join.

An element p in a lattice $\mathcal{P}$ is *join-irreducible* if it cannot be written as the join of other elements of P, i.e. if we can write $p = p_1 \vee p_2 \vee \cdots \vee p_s$ for $p_1,p_2,\ldots,p_s \in P$, then there is some $i \in \{1,2,\ldots,s\}$ with $p = p_i$. We denote the set of join-irreducible elements of $\mathcal{P}$ by $\mathcal{J}(\mathcal{P})$. In particular, $p \in \mathcal{J}(\mathcal{P})$ if and only if p covers exactly one element, which we then denote by $p_\star$. Dually, p is called *meet-irreducible* if it cannot be written as the meet of other elements of P, and we denote the set of meet-irreducible elements of $\mathcal{P}$ by $\mathcal{M}(\mathcal{P})$. In particular, $p \in \mathcal{M}(\mathcal{P})$ if and only if p is covered by exactly one element, which we then denote by $p^\star$. According to **[76]** we say that a lattice $\mathcal{P}$ is *extremal* if $\lvert\mathcal{J}(\mathcal{P})\rvert = \ell(\mathcal{P}) = \lvert\mathcal{M}(\mathcal{P})\rvert$.

Let us introduce some further lattice-theoretic concepts. An element p is called *left-modular* if it satisfies the following equality for all $q,q' \in P$ with $q < q'$:

$$(q \vee p) \wedge q' = q \vee (p \wedge q'). \tag{1.1}$$

If there exists a maximal chain in $\mathcal{P}$ consisting of left-modular elements, then we say that $\mathcal{P}$ is *left-modular*. The following characterization of left-modular elements turns out to be very useful.

THEOREM 1.1.21 ([75, Theorem 1.4])

Let $\mathcal{P} = (P, \leq)$ be a finite lattice, and let $p \in P$. The following are equivalent:
(i) the element p is left-modular; and
(ii) for any $q, q' \in P$ with $q \lessdot q'$, we have $p \wedge q = p \wedge q'$ or $p \vee q = p \vee q'$ but not both.

Now suppose that $\mathcal{P}$ is a left-modular lattice, and let $\hat{0} = p_0 \lessdot p_1 \lessdot \cdots \lessdot p_s = \hat{1}$ be a maximal chain of $\mathcal{P}$ consisting of left-modular elements. We define an edge-labeling of $\mathcal{P}$ by

$$\psi(q, q') = \min\{i \in \{1, 2, \ldots, s\} \mid q \vee p_i \wedge q' = q'\}, \tag{1.2}$$

for all $q, q' \in P$ with $q \lessdot q'$. We have the following result.

THEOREM 1.1.22 ([74])

If $\mathcal{P}$ is a left-modular lattice with a distinguished left-modular chain, then the edge-labeling in (1.2) is an EL-labeling. In particular, every left-modular lattice is EL-shellable.

For more results on left-modular lattices, including a generalization to posets, we refer the reader to [27,75,79,121]. Recall further that $\mathcal{P}$ is called *distributive* if it satisfies

$$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r) \quad \text{and}, \tag{1.3}$$
$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r), \tag{1.4}$$

for all $p, q, r \in P$. Following [121], a *trim lattice* is a lattice that is extremal and left-modular. It follows from [76] that an interval of an extremal lattice need not necessarily be extremal again. Trim lattices, however, behave nicely with respect to this property.

THEOREM 1.1.23 ([121, Theorem 1])

Every interval of a trim lattice is trim again.

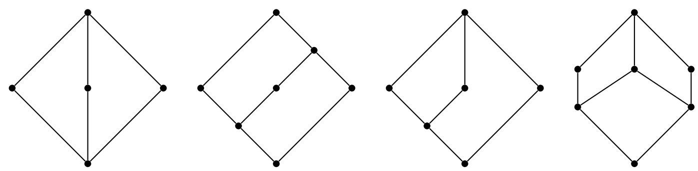
We can thus consider trimness as a generalization of distributivity to ungraded lattices, since every graded trim lattice is distributive, see [76, Theorem 17]. Moreover, trim lattices have a nice topological structure. Recall that a lattice $\mathcal{P}$ is *nuclear* if $\hat{1}$ is the join of the atoms of $\mathcal{P}$.

THEOREM 1.1.24 ([121, Theorem 7])

Let $\mathcal{P}$ be a trim lattice with k atoms. If $\mathcal{P}$ is nuclear, then the order complex $\Delta(\overline{\mathcal{P}})$ is homotopy equivalent to a sphere of dimension $k-2$. Otherwise $\Delta(\overline{\mathcal{P}})$ is contractible. In particular, the Möbius function of $\mathcal{P}$ takes values only in $\{-1, 0, 1\}$.

There is a weakening of the distributive laws, in the sense that whenever some element $p \in P$ has the same join (or dually, meet) with two elements $q, r \in P$, then the corresponding distributive law can be applied to the triple (p, q, r). Formally we say that a lattice $\mathcal{P}$ is *semidistributive* if it satisfies

$$p \vee q = p \vee r \quad \text{implies} \quad p \vee q = p \vee (q \wedge r), \quad \text{and} \tag{1.5}$$
$$p \wedge q = p \wedge r \quad \text{implies} \quad p \wedge q = p \wedge (q \vee r), \tag{1.6}$$



(a) $\mathcal{L}_1$. (b) $\mathcal{L}_2$. (c) $\mathcal{L}_3$. (d) $\mathcal{L}_4$.

FIGURE 3. Essentially the forbidden sublattices of a semidistributive lattice.

for all $p, q, r \in P$. If $\mathcal{P}$ satisfies only (1.5) or (1.6), then we call $\mathcal{P}$ *join-semidistributive* or *meet-semidistributive*, respectively. There is a nice characterization of semidistributive lattices in terms of forbidden sublattices.

THEOREM 1.1.25 ([41])

A lattice with no infinite chains is semidistributive if and only if it contains no sublattice isomorphic to the lattices (or their duals) shown in Figure 3.

The next lemma states that semidistributive lattices have the same number of join- and meet-irreducible elements.

LEMMA 1.1.26 ([43])

If $\mathcal{P}$ is semidistributive, then $|\mathcal{J}(\mathcal{P})| = |\mathcal{M}(\mathcal{P})|$.

1.1.6. Lattices as Algebraic Structures. Besides the poset-theoretic approach, we can also define lattices in an algebraic way. Consider a set L equipped with two binary operations $\vee : L \times L \to L$ and $\wedge : L \times L \to L$ and two constants $\hat{0}, \hat{1} \in L$ having the following five properties:

(i) if $x \in L$, then $x \vee x = x = x \wedge x$; (IDEMPOTENCE)
(ii) if $x, y \in L$, then $x \vee (x \wedge y) = x = x \wedge (x \vee y)$; (ABSORPTION)
(iii) if $x, y \in L$, then $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$; (COMMUTATIVITY)
(iv) if $x, y, z \in L$, then $x \vee (y \vee z) = (x \vee y) \vee z$ and $x \wedge (y \wedge z) = (x \wedge y) \wedge z$; (ASSOCIATIVITY)
(v) if $x \in L$, then $x \vee \hat{0} = x = x \wedge \hat{1}$. (IDENTITY)

In this case, we call the algebraic structure $(L; \vee, \wedge, \hat{0}, \hat{1})$ a *lattice*. It is checked easily that in a bounded poset $(P, \leq)$ that is a lattice, the operations join and meet and the elements $\hat{0}$ and $\hat{1}$ satisfy the above axioms. On the other hand, if an algebraic structure $(L; \vee, \wedge, \hat{0}, \hat{1})$ is a lattice, then we can define a partial order on L via $x \leq y$ if and only if $x \vee y = y$ (if and only if $x \wedge y = x$), and we can easily check that the bounded poset $(L, \leq)$ is indeed a lattice. In view of this equivalence of concepts, we will not distinguish between the order-theoretic or the algebraic definition of a lattice, and use both concepts interchangeably.

Viewing lattices as algebraic structures has certain advantages. For instance it allows for a natural notion of homomorphisms. A *lattice homomorphism* is a map $f : K \to L$ from a lattice $\mathcal{K} = (K; \vee_K, \wedge_K, \hat{0}_K, \hat{1}_K)$ to a lattice $\mathcal{L} = (L; \vee_L, \wedge_L, \hat{0}_L, \hat{1}_L)$ that preserves the lattice operations, i.e.

$$f(x \vee_K y) = f(x) \vee_L f(y) \quad \text{and} \quad f(x \wedge_K y) = f(x) \wedge_L f(y)$$

for all $x, y \in K$. A lattice homomorphism is called *bounded* if for every $x \in L$ the preimage $f^{-1}(x)$ is a bounded subposet of $(K, \leq_K)$. A subset $X \subseteq L$ is said to *generate* $\mathcal{L}$ if every

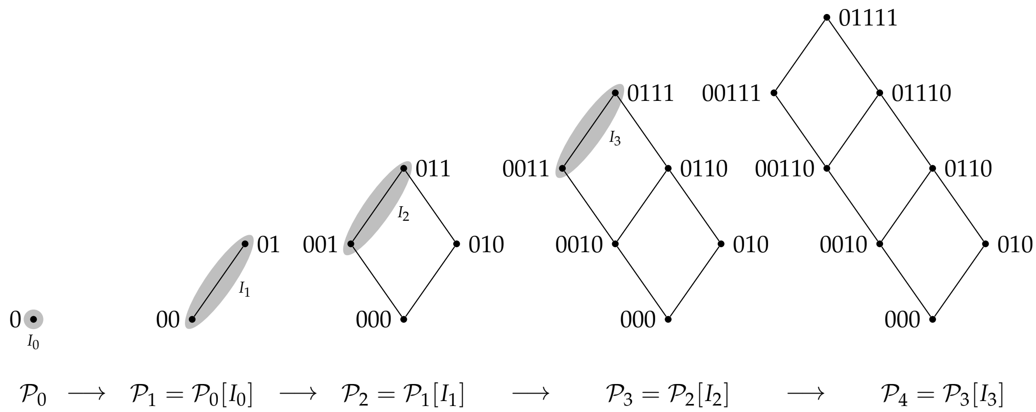


FIGURE 4. The lattice $\mathcal{D}_{24}$ can be constructed from the one-element lattice by successive doubling of intervals. (We have omitted parentheses in the diagram labels for brevity.)

$x \in L$ can be written as a sequence of joins and meets of elements in X. Moreover, a lattice $\mathcal{F} = (F; \vee, \wedge, \hat{0}, \hat{1})$ that is generated by X is called the *free lattice over X* if every map from X to some lattice $\mathcal{L}$ can be extended uniquely to a lattice homomorphism from $\mathcal{F}$ to $\mathcal{L}$. (In other words, $\mathcal{F}$ is the free object on X in the category of lattices.) Finally we call a lattice $\mathcal{L}$ a *bounded-homomorphic image of a free lattice* if there exists a bounded lattice homomorphism from some free lattice to $\mathcal{L}$.

Bounded-homomorphic images of free lattices have two nice properties. On the one hand they are semidistributive, and on the other hand they can be constructed in a very nice way from the one-element lattice **1**.

THEOREM 1.1.27 ([55, Theorem 2.20])

If $\mathcal{P}$ is a bounded-homomorphic image of a free lattice, then $\mathcal{P}$ is semidistributive.

Now let $\mathcal{P} = (P, \leq)$ be a lattice, and let I be a closed interval of $\mathcal{P}$. Define the lattice $\mathcal{P}[I] = (P', \leq')$ by $P' = (P \setminus I) \cup \big(I \times \{0,1\}\big)$, and

$$p \leq' q = \begin{cases} p \leq q, & \text{if } p, q \in P \setminus I,\\ p \leq q', & \text{if } p \in P \setminus I, \text{ and } q = (q', i) \text{ for } q' \in I \text{ and } i \in \{0,1\},\\ p' \leq q, & \text{if } p = (p', i) \text{ for } p' \in I \text{ and } i \in \{0,1\}, \text{ and } q \in P \setminus I,\\ p' \leq q', & \text{if } p = (p', i), q = (q', j) \text{ for } p', q' \in I \text{ and } i, j \in \{0,1\} \text{ with } i \leq j. \end{cases}$$

We say that $\mathcal{P}[I]$ arises from $\mathcal{P}$ by *doubling the interval I*. Figure 4 indicates how the lattice $\mathcal{D}_{24}$ can be constructed from the one-element lattice by successively doubling intervals.

THEOREM 1.1.28 ([43, Theorem 5.1])

A lattice $\mathcal{P}$ is a bounded-homomorphic image of a free lattice if and only if there exists a sequence $\mathbf{1} = \mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_t = \mathcal{P}$, and a sequence of intervals $I_0, I_1, \ldots, I_{t-1}$ such that I_s is a closed interval of $\mathcal{P}_s$, and $\mathcal{P}_{s+1} = \mathcal{P}_s[I_s]$ for all $s \in \{0, 1, \ldots, t-1\}$.

Another lattice property that implies semidistributivity is the existence of a so-called canonical join-representation. Let $\mathcal{P} = (P, \leq)$ be a lattice, and let $p \in P$. A set $Z =$

$\{z_1, z_2 \ldots, z_s\} \subseteq P$ is called a *join-representation of p in $\mathcal{P}$* if $p = z_1 \vee z_2 \vee \cdots \vee z_s$. For two join-representations Z and Z' of p in $\mathcal{P}$ we say that *Z refines Z'* if for every $z \in Z$ there exists a $z' \in Z'$ with $z \leq z'$. A join-representation Z of p in $\mathcal{P}$ is called *canonical* if it refines every other join-representation of p in $\mathcal{P}$. In this case we usually write Z_p instead of Z. It follows immediately that Z_p is an antichain in $\mathcal{P}$, i.e. the elements of Z_p are pairwise incomparable, and each element in Z_p is join-irreducible.

LEMMA 1.1.29 ([**55**, Lemma 2.22])

If every element of a lattice $\mathcal{P}$ has a canonical join-representation, then $\mathcal{P}$ is join-semidistributive.

EXAMPLE 1.1.30

Let us again consider the lattice $\mathcal{D}_{24}$ from Figure 1(a). We have $12 = 4 \vee 6 = 3 \vee 4$. Thus both $\{4, 6\}$ and $\{3, 4\}$ are join-representations of 12. Since $3 \mid 6$, it follows that $\{3, 4\}$ refines $\{4, 6\}$. Moreover, it can be checked easily that $\{3, 4\}$ is the canonical join-representation of 12 in $\mathcal{D}_{24}$.

1.2. Coxeter Groups

The posets we investigate in this thesis are intrinsically connected to Coxeter groups, so we will introduce the necessary concepts here. Coxeter groups play an important role in algebra, geometry and combinatorics, since they can be seen as a natural generalization of the symmetric group. Moreover, Coxeter groups can be viewed as groups generated by reflections, and in this guise they generalize the Weyl groups. Even though we will mainly use the algebraic representation of Coxeter groups, we will elaborate on this so-called reflection representation in more detail in Section 1.2.3. We will use this aspect of Coxeter group theory also in Chapter 4, where we consider the posets of generalized noncrossing partitions.

A good introduction to Coxeter groups from an algebraic point of view is [**23**], while [**63**] focuses more on geometric aspects of the reflection representation of Coxeter groups.

1.2.1. Coxeter Systems. A *Coxeter matrix* is a symmetric $(n \times n)$-matrix $(m_{i,j})_{1 \leq i,j \leq n}$ with the property that $m_{i,i} = 1$ for $i \in \{1, 2, \ldots, n\}$ and $m_{i,j} \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ for $i, j \in \{1, 2, \ldots, n\}$ and $i < j$. With each Coxeter matrix we can associate a group given by the presentation

$$W = \langle s_1, s_2, \ldots, s_n \mid (s_i s_j)^{m_{i,j}} = \varepsilon \rangle, \tag{1.7}$$

where ε denotes the identity of W. We call such a group a *Coxeter group*, and if we abbreviate $S = \{s_1, s_2, \ldots, s_n\}$, then we call the pair (W, S) a *Coxeter system*. The elements of S are called the *Coxeter generators of W*, and the cardinality of S is called the *rank of (W, S)*.

If we can write S as a disjoint union of two nontrivial sets S_1 and S_2 such that each element of S_1 commutes with every element of S_2, then we say that W is *reducible*, and we have $W \cong W_1 \times W_2$ where W_1 and W_2 are the Coxeter groups generated by S_1 and S_2, respectively. If this is not the case, then we call W *irreducible*.

EXAMPLE 1.2.1

Perhaps the simplest example of a Coxeter group is the dihedral group of order $2k$, i.e. the group of symmetries of a regular k-gon Δ_k. We denote this group by $I_2(k)$, and this name comes from the classification of finite irreducible Coxeter groups, see Theorem 1.2.5. It is generated naturally by two elements: a reflection s through a symmetry axis of Δ_k and a rotation r by an angle of $\frac{2\pi}{k}$. By definition s has order 2, r has order k, and we can quickly verify that $rs = sr^{-1}$.

However, for our point of view, the following choice of generators of $I_2(k)$ is more suitable: it can be checked quickly that r can be expressed as a concatenation of two reflections through symmetry axes which enclose an angle of $\frac{\pi}{k}$, say $r = s_1 s_2$. Thus in view of the previous paragraph, it follows that the product $s_1 s_2$ has order k and that both s_1 and s_2 have order two. It can be checked that these are the only relations in $I_2(k)$. Hence $I_2(k)$ can be represented by the Coxeter matrix

$$M_{I_2(k)} = \begin{pmatrix} 1 & k \\ k & 1 \end{pmatrix},$$

and it is thus a Coxeter group of rank 2.

Example 1.2.2

Another classical example of a Coxeter group is the symmetric group on $\{1, 2, \ldots, n\}$, which we will denote by A_{n-1}. (Again this name comes from the classification.) Recall that A_{n-1} is generated naturally by the set of adjacent transpositions, i.e. by $s_i = (i\ i{+}1)$ for $i \in \{1, 2, \ldots, n-1\}$. It follows immediately that the product $s_i s_j$ has order three if and only if $j = i + 1$ and it has order two otherwise. It can be checked that these are the only relations in A_{n-1}. Hence A_{n-1} can be represented by the $\big((n-1) \times (n-1)\big)$-Coxeter matrix

$$M_{A_{n-1}} = \begin{pmatrix} 1 & 3 & 2 & \cdots & 2 & 2 & 2 \\ 3 & 1 & 3 & \cdots & 2 & 2 & 2 \\ 2 & 3 & 1 & \cdots & 2 & 2 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 2 & 2 & 2 & \cdots & 1 & 3 & 2 \\ 2 & 2 & 2 & \cdots & 3 & 1 & 3 \\ 2 & 2 & 2 & \cdots & 2 & 3 & 1 \end{pmatrix},$$

and it is thus a Coxeter group of rank $n - 1$. (This explains why we denote the symmetric group on $\{1, 2, \ldots, n\}$ by A_{n-1}.)

Remark 1.2.3

We remark that a group might have different, nonisomorphic presentations as a Coxeter group. Hence it is sometimes important to specify the corresponding Coxeter system explicitly.

For example, consider the dihedral group $I_2(6)$. We have seen in Example 1.2.1 that $I_2(6)$ is the group of symmetries of a regular hexagon, and it thus admits an irreducible Coxeter presentation of the form

$$I_2(6) = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^6 = \varepsilon \rangle.$$

See Figure 5(a) for an illustration of the symmetries. We can also consider $I_2(6)$ as a group acting on two triangles. From this point of view we exhibit another presentation, which is, however, reducible:

$$I_2(6) = \langle r, r_1, r_2 \mid r^2 = r_1^2 = r_2^2 = (rr_1)^2 = (rr_2)^2 = (r_1 r_2)^3 = \varepsilon \rangle,$$

(a) Coxeter symmetries of a regular hexagon.

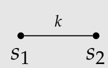
(b) Coxeter symmetries of the star of David.

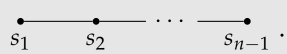
FIGURE 5. Two Coxeter presentations of $I_2(6)$.

where r is the reflection that maps the one triangle to the other triangle, and r_1 and r_2 are the Coxeter generators of the dihedral group $I_2(3)$ considered as the symmetry group of a triangle. See Figure 5(b) for an illustration.

1.2.2. Coxeter Diagrams. From now on let (W, S) be a Coxeter system of rank n. There is a very simple and intuitive way to graphically represent (W, S): a *Coxeter diagram* $\Gamma_{(W,S)}$ is a labeled graph with vertices $v_{s_1}, v_{s_2}, \ldots, v_{s_n}$, where there is an edge between v_{s_i} and v_{s_j} if and only if $m_{i,j} \geq 3$. Moreover, such an edge is labeled by $m_{i,j}$ if and only if $m_{i,j} \geq 4$.

EXAMPLE 1.2.4

The Coxeter diagram of the dihedral group $I_2(k)$ is 

, while the Coxeter diagram of the symmetric group A_{n-1} is 

.

It follows immediately from the definition that a Coxeter diagram is connected if and only if the corresponding Coxeter system is irreducible. With the help of these Coxeter diagrams, COXETER was able to characterize the finite irreducible Coxeter groups.

THEOREM 1.2.5 ([40, Theorem ‡])

An irreducible Coxeter group is finite if and only if its Coxeter diagram is one of the diagrams in Figure 6.

DISCLAIMER 1.2.6

Whenever we consider a finite irreducible Coxeter group W, then we implicitly use the Coxeter system (W, S) such that the corresponding Coxeter diagram is shown in Figure 6.

The impact of COXETER's classification becomes even clearer, in view of the following interpretation of Coxeter groups as groups generated by reflections.

1.2.3. Reflection Representation. So far we have defined Coxeter groups in a purely algebraic way. However, they have a very natural geometric interpretation that we will describe next, following [63, Chapter 5.3]. Let (W, S) be a Coxeter system of rank n, and consider the n-dimensional real vector space V spanned by $\{\mathbf{v}_s \mid s \in S\}$. That is, we take some basis of V

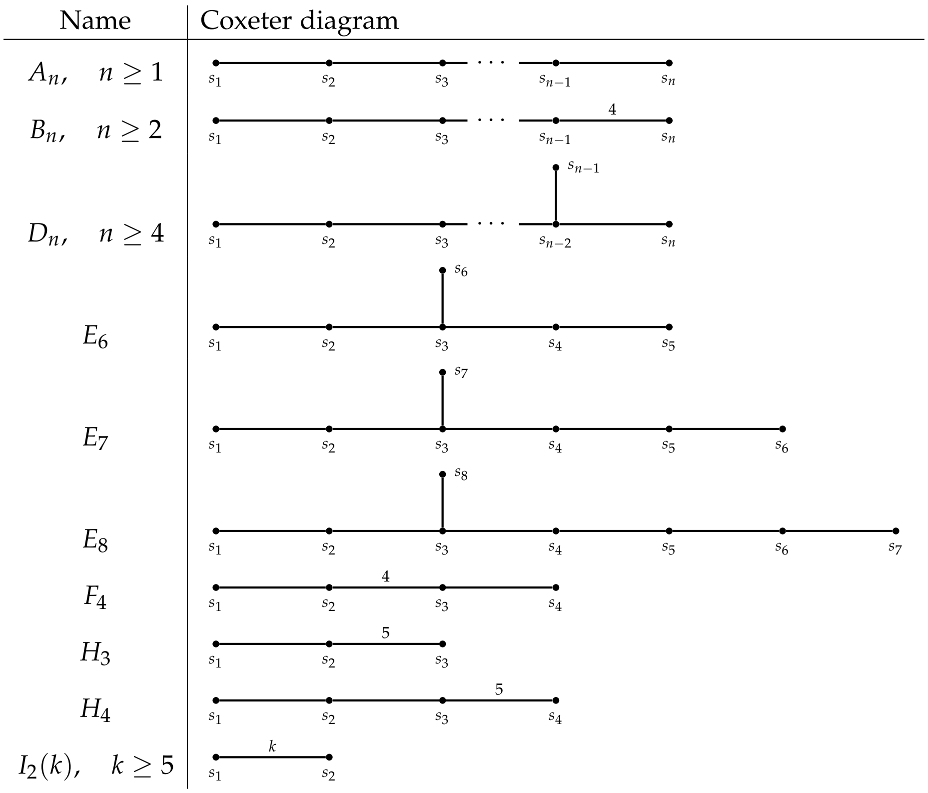


FIGURE 6. The Coxeter diagrams of the finite irreducible Coxeter groups.

and associate each basis vector with one Coxeter generator. The crucial step in understanding the geometry of W is the definition of a symmetric bilinear form that is given on the basis of V by

$$B(\mathbf{v}_{s_i}, \mathbf{v}_{s_j}) = -\cos\left(\frac{\pi}{m_{i,j}}\right), \tag{1.8}$$

and we set $B(\mathbf{v}_{s_i}, \mathbf{v}_{s_j}) = -1$ if $m_{i,j} = \infty$. This bilinear form allows for the concept of orthogonality in V, by saying that two vectors $\mathbf{v}, \mathbf{v}' \in V$ are orthogonal if and only of $B(\mathbf{v}, \mathbf{v}') = 0$. It follows from (1.8) that $B(\mathbf{v}_s, \mathbf{v}_s) = 1$ and thus that the orthogonal complement of $\mathbf{v}_s$ in V is a hyperplane, which we will denote by H_s. For $s \in S$ we define the *reflection σ_s through H_s* to be the following linear transformation on V:

$$\sigma_s(\mathbf{v}) = \mathbf{v} - 2B(\mathbf{v}_s, \mathbf{v})\mathbf{v}_s,$$

for $\mathbf{v} \in V$. Then, σ_s fixes H_s pointwise and sends $\mathbf{v}_s$ to its negative. (In fact, these two properties can be taken as a definition for a reflection. In the case where $B(\cdot,\cdot)$ is an inner product, this coincides with the usual definition.) The hyperplane H_s is called the *reflection hyperplane of σ_s*. In particular, the bilinear form $B(\cdot,\cdot)$ is defined in such a way that the dihedral angle between the reflection hyperplanes H_{s_i} and H_{s_j} is $\frac{2\pi}{m_{i,j}}$. Moreover, the bilinear form $B(\cdot,\cdot)$

is preserved under the action of σ_s, since $B(\mathbf{v}_s, \mathbf{v}_s) = 1$ implies

$$\begin{aligned}B(\sigma_s\mathbf{v}, \sigma_s\mathbf{v}') &= B\big(\mathbf{v} - 2B(\mathbf{v}_s, \mathbf{v})\mathbf{v}_s, \mathbf{v}' - 2B(\mathbf{v}_s, \mathbf{v}')\mathbf{v}_s\big)\\ &= B\big(\mathbf{v}, \mathbf{v}' - 2B(\mathbf{v}_s, \mathbf{v}')\mathbf{v}_s\big) - B\big(2B(\mathbf{v}_s, \mathbf{v})\mathbf{v}_s, \mathbf{v}' - 2B(\mathbf{v}_s, \mathbf{v}')\mathbf{v}_s\big)\\ &= B(\mathbf{v}, \mathbf{v}') - 2B(\mathbf{v}_s, \mathbf{v}')B(\mathbf{v}, \mathbf{v}_s) - 2B(\mathbf{v}_s, \mathbf{v})B(\mathbf{v}_s, \mathbf{v}') + 4B(\mathbf{v}_s, \mathbf{v})B(\mathbf{v}_s, \mathbf{v}')B(\mathbf{v}_s, \mathbf{v}_s)\\ &= B(\mathbf{v}, \mathbf{v}').\end{aligned}$$

Let G be the subgroup of $GL(V)$ generated by the reflections σ_s for $s \in S$. In view of the previous reasoning it follows that G preserves $B(\cdot, \cdot)$ as well, and we have the following result.

THEOREM 1.2.7 ([**63**, Theorem 5.3 and Corollary 5.4])

There exists a unique isomorphism σ between W and G that sends s to σ_s for all $s \in S$.

We call the isomorphism σ from Theorem 1.2.7 the *geometric representation of W*, and this theorem states that σ is a faithful representation of W in $GL(V)$. The bilinear form $B(\cdot, \cdot)$ is not necessarily an inner product on V. However, the following theorem characterizes the situation in which this is the case.

THEOREM 1.2.8 ([**63**, Theorem 6.4])

Let (W, S) be a Coxeter system. Then, W is finite if and only if the linear form $B(\cdot, \cdot)$ from (1.8) is positive definite.

This implies together with Theorems 1.2.5 and 1.2.7 that every finite Coxeter group is a finite reflection group, and that the rank of a finite Coxeter group corresponds to the dimension of the vector space on which this group acts essentially as a reflection group. On the other hand [**63**, Theorem 1.9] implies that every finite reflection group admits a presentation as a finite Coxeter group.

Among the finite reflection groups there is a subclass of great importance: the *finite Weyl groups*. These are reflection groups for which the Coxeter matrix has entries only in $\{1, 2, 3, 4, 6\}$. Each finite reflection group is completely characterized by a so-called root system, which is a certain set of vectors in a Euclidean vector space. (For an exact definition, see for instance [**63**, Section 1.2].) It suffices here to say that the roots are vectors orthogonal to the reflection hyperplanes of the corresponding reflection group. These root systems were used by WEYL to characterize the semisimple Lie algebras, see [**128–130**]. This characterization was simplified by DYNKIN in [**46**], who used diagrams similar to the Coxeter diagrams in order to represent the root systems of the the finite irreducible Weyl groups.

We conclude this section with the observation that the reflections corresponding to the Coxeter generators are not the only reflections in a reflection group. It follows from linear algebra that for $w \in W$ and $s \in S$ the transformation $\sigma(w^{-1}sw)$ fixes the hyperplane $w^{-1}(H_s)$ pointwise and sends $w^{-1}(\mathbf{v}_s)$ to its negative. Thus $\sigma(w^{-1}sw)$ acts as a reflection on V, and we define

$$T = \{w^{-1}sw \mid w \in W, S \in S\}. \tag{1.9}$$

It follows from [**63**, Proposition 1.14] that T is the set of all reflections of W. Clearly S is a distinguished subset of T. Even though we mainly consider Coxeter groups from an algebraic point of view, we refer to the elements of S as the *simple reflections of W* and to the elements of T as the *reflections of W*.

REMARK 1.2.9

In this section we have considered only groups generated by reflections in a real vector space. Under a suitable relaxation of the concept of a reflection, this can be generalized to complex vector spaces, and the resulting groups are complex reflection groups. We will return to this generalization in Chapter 4, where we also formally define these groups.

1.2.4. Reduced Decompositions and Length. Now we return to the algebraic nature of the Coxeter groups. Since S is a generating set of W, every group element $w \in W$ can be written as a product of the simple reflections of W. This gives rise to a length function, which we will call the *Coxeter length*, in the following way:

$$\ell_S : W \to \mathbb{N}, \quad w \mapsto \min\{k \mid w = s_{i_1}s_{i_2}\cdots s_{i_k}, s_{i_j} \in S \text{ for } 1 \leq j \leq k\}. \tag{1.10}$$

If $\ell_S(w) = k$, then we call every word $w = s_{i_1}s_{i_2}\cdots s_{i_k}$ a *reduced decomposition of w*. An element $w \in W$ is a *Coxeter element of (W, S)* if it has a reduced decomposition in which every simple reflection appears exactly once, i.e. if we can write $w = s_{\sigma(1)}s_{\sigma(2)}\cdots s_{\sigma(n)}$ for some permutation σ of $\{1, 2, \ldots, n\}$.

EXAMPLE 1.2.10

Let us consider the symmetric group A_{n-1} again. The simple reflections of A_{n-1} are precisely the adjacent transpositions. It is well known that every permutation of $\{1, 2, \ldots, n\}$ can be decomposed into a product of adjacent transpositions, and the minimal length of such a decomposition corresponds to the Coxeter length of A_{n-1}. Moreover, every long cycle can be decomposed into a product of $n - 1$ adjacent transpositions, none of which occurs twice. Hence each long cycle is a Coxeter element of A_{n-1}.

According to [105] the Coxeter elements of (W, S) have an interesting connection to the Coxeter diagram $\Gamma_{(W,S)}$. An *orientation of $\Gamma_{(W,S)}$* is an assignment of "directions" to the edges of $\Gamma_{(W,S)}$, i.e. for an edge between two generators s_i and s_j we fix whether this edge runs from s_i to s_j or in the opposite direction. An *acyclic orientation of $\Gamma_{(W,S)}$* is an orientation of $\Gamma_{(W,S)}$ that does not contain oriented cycles. Given an acyclic orientation $\vec{\Gamma}_{(W,S)}$ of the Coxeter graph of (W, S) we can define an element $\vec{\gamma} \in W$ with the following property: whenever an edge is oriented from s_i to s_j, then we require that the letter s_i occurs before the letter s_j in every reduced decomposition of $\vec{\gamma}$. Theorem 1.5 in [105] states that $\vec{\gamma}$ is a Coxeter element of (W, S) and that this correspondence is one-to-one. This connection plays a crucial role in the original definition of the Cambrian lattices due to READING, see [93]. We define the Cambrian lattices in detail in Section 3.2.

EXAMPLE 1.2.11

Consider the group $W = A_5$ together with the orientation $s_1 \rightarrow s_2 \leftarrow s_3 \rightarrow s_4 \leftarrow s_5$. The corresponding Coxeter element is $\gamma = s_1s_3s_5s_2s_4 = (1\ 2\ 4\ 6\ 5\ 3)$, and we can easily check that in each reduced decomposition of γ, the letter s_1 appears before s_2, the letter s_3 appears before both s_2 and s_4, and the letter s_5 appears before s_4. In fact, we can write $\gamma = \gamma_1\gamma_2$, where $\gamma_1 = s_1s_3s_5$ and $\gamma_2 = s_2s_4$, and each letter from γ_1 commutes with every letter from γ_2. Coxeter elements that possess such a reduced decomposition are called *bipartite*.

We have seen in the end of Section 1.2.3 that the set T of all reflections can also be taken as a generating set of W. Hence we can define reduced decompositions of the elements of W

in terms of all reflections as well as a length function analogously as before. More precisely, we define the *absolute length* by

$$\ell_T : W \to \mathbb{N}, \quad w \mapsto \min\{k \mid w = t_{i_1}t_{i_2}\cdots t_{i_k}, t_{i_j} \in T \text{ for } 1 \leq j \leq k\}, \tag{1.11}$$

and if $\ell_T(w) = k$, then we call every word $w = t_{i_1}t_{i_2}\cdots t_{i_k}$ a *reduced T-decomposition of w*. Since $S \subseteq T$ it follows immediately that $\ell_T(w) \leq \ell_S(w)$ for all $w \in W$.

EXAMPLE 1.2.12

We have seen in Example 1.2.2 that the simple reflections of A_{n-1} correspond to the adjacent transpositions. It follows by definition of T that the reflections of A_{n-1} are precisely *all* transpositions.

If we consider for instance the permutation $\pi = (1\ 4\ 3)(2\ 6\ 5) \in A_5$, then we can quickly check that

$$\pi = (1\,3)(1\,4)(2\,6)(5\,6) = (3\,4)(2\,3)(1\,2)(5\,6)(4\,5)(3\,4)(4\,5)(2\,3).$$

Thus we have $\ell_T(\pi) = 4$ and $\ell_S(\pi) = 8$.

1.2.5. Partial Orders on Coxeter Groups. In this section we will introduce the two main partial orders on Coxeter groups that we need in this thesis. We start with a partial order that is defined in terms of the simple reflections of (W, S).

DEFINITION 1.2.13

Let (W, S) be a Coxeter system. The *(right) weak order on* (W, S) is the partial order defined by

$$u \leq_S v \quad \text{if and only if} \quad \ell_S(v) = \ell_S(u) + \ell_S(u^{-1}v),$$

for all $u, v \in W$. We will usually write $\mathcal{W} = (W, \leq_S)$ for the corresponding poset.

Figure 7 shows the right weak order on the symmetric group A_3.

REMARK 1.2.14

We can also define a left weak order on W analogously to Definition 1.2.13, which does not coincide with the right weak order, but is isomorphic via the map $w \mapsto w^{-1}$. Hence we will usually omit the qualifier "right" and speak of the weak order only.

The poset $\mathcal{W}$ has several nice properties.

PROPOSITION 1.2.15 ([23, Proposition 3.1.6])

For every $u, v \in W$ with $u \leq_S v$, we have $[u, v] \cong [\varepsilon, u^{-1}v]$ via the poset isomorphism $w \mapsto u^{-1}w$.

PROPOSITION 1.2.16 ([97, Proposition 2.19])

For every $w \in W$ the interval $[\varepsilon, w]$ in $\mathcal{W}$ is isomorphic to the dual of the interval $[\varepsilon, w^{-1}]$.

THEOREM 1.2.17 ([23, Proposition 3.1.2 and Theorem 3.2.1])

The poset $\mathcal{W}$ is a graded meet-semilattice with rank function ℓ_S.

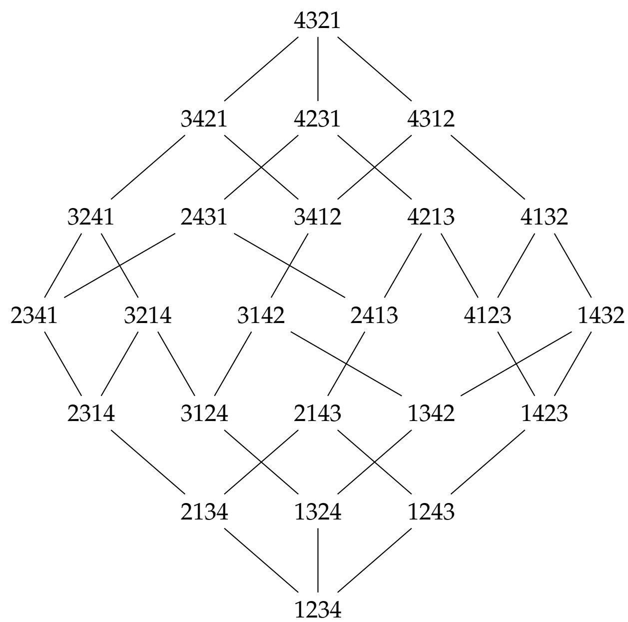


FIGURE 7. The weak order on the symmetric group A_3.

If W is finite, then there exists a unique element $w_o \in W$ of maximal length, and in particular we have $\ell_S(w_o) = |T|$, see [**23**, Proposition 2.3.2(iv)]. Proposition 1.2.16 implies that w_o is an involution and that $\mathcal{W}$ is a self-dual lattice if and only if W is finite. Moreover, Proposition 1.2.15 implies that it is sufficient to understand the intervals of the form $[\varepsilon, w]$ for some $w \in W$. But more can be said. Recall that a poset $(P, \leq)$ with a least element $\hat{0}$ is called *finitary* if for every $p \in P$ the interval $[\hat{0}, p]$ is finite. The following proposition is well known to the community, however we could not find an explicit reference for it. Thus we give a simple proof here.

PROPOSITION 1.2.18

The semilattice $\mathcal{W}$ is finitary.

PROOF. Let W be a Coxeter group of rank n, and let $w \in W$ with $\ell_S(w) = k$. We want to show that the interval $[\varepsilon, w]$ is finite. Recall for instance from [**23**, Proposition 3.1.2(i)], that the maximal chains in $[\varepsilon, w]$ are in bijection with the reduced decompositions of w. Since w has length k, there are at most n^k reduced decompositions of w, which is clearly a finite number. Hence there is only a finite number of maximal chains in $[\varepsilon, w]$ and each of these chains is finite. Thus the interval itself has to be finite. $\square$

We will often find the following equivalent definition of $\leq_S$ convenient. We say that $t \in T$ is a *(left) inversion of $w \in W$* if $\ell_S(tw) < \ell_S(w)$. If we write $\mathrm{inv}(w)$ for the set of all inversions of w, then it follows from [**23**, Corollary 1.4.5] that $\ell_S(w) = |\mathrm{inv}(w)|$.

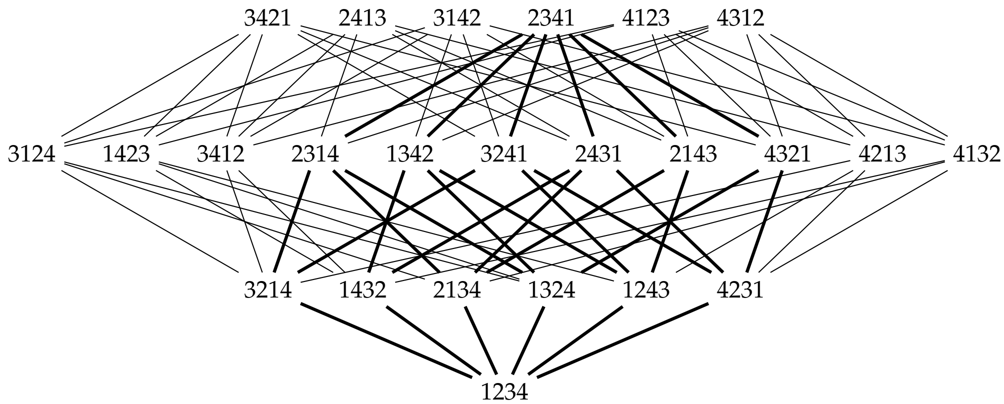


FIGURE 8. The absolute order on the symmetric group A_3. The highlighted interval is a lattice of noncrossing partitions.

PROPOSITION 1.2.19 ([23, Proposition 3.1.3])

Let (W,S) be a Coxeter system. For every $u,v \in W$ we have

$$u \leq_S v \quad \textit{if and only if} \quad \mathrm{inv}(u) \subseteq \mathrm{inv}(v).$$

For $s \in S$ let $W_{\geq s} = \{w \in W \mid s \leq_S w\}$, and let $W_{\not\geq s} = \{w \in W \mid s \not\leq_S w\}$.

PROPOSITION 1.2.20 ([97, Proposition 2.18])

Let $w \in W$, and let $s \in S$. Then, $\ell_S(sw) < \ell_S(w)$ if and only if $s \leq_S w$ if and only if $s \in \mathrm{inv}(w)$. Left multiplication by s is a poset isomorphism from $(W_{\not\geq s}, \leq_S)$ to $(W_{\geq s}, \leq_S)$. If $w \lessdot_S w', s \leq_S w'$ and $s \not\leq_S w$, then $w' = sw$.

There exists a distinguished subset of the inversions of w, the so-called *cover reflections of w*. These are the inversions $t \in \mathrm{inv}(w)$ for which there exists some $s \in S$ such that $tw = ws$. We denote the set of cover reflections of w by $\mathrm{cov}(w)$. In particular, if $t \in \mathrm{cov}(w)$, then $tw \lessdot_S w$, and we have $\mathrm{inv}(tw) = \mathrm{inv}(w) \setminus \{t\}$.

The second partial order on Coxeter groups that we consider in this thesis is defined very similarly to the weak order. We simply replace the set of simple reflections by the set of all reflections, and this partial order was first considered in [33].

DEFINITION 1.2.21

Let W be a Coxeter group, and let T denote the set of all reflections of W. The *absolute order on W* is the partial order defined by

$$u \leq_T v \quad \text{if and only if} \quad \ell_T(v) = \ell_T(u) + \ell_T(u^{-1}v),$$

for all $u,v \in W$.

Figure 8 shows the poset $(A_3, \leq_T)$. In contrast to the weak order, the poset $(W, \leq_T)$ is not bounded and not a semilattice. However, it is a graded poset with rank function ℓ_T, and if W is finite, then it contains a very well-behaved subposet: if $\gamma \in W$ is a Coxeter element, then the interval $[\varepsilon, \gamma]$ in $(W, \leq_T)$ is called the *lattice of W-noncrossing partitions*. We study this lattice in Chapter 4, and we postpone the statement of its basic properties until then.

CHAPTER 2

The m-Tamari Lattices

2.1. Introduction

Perhaps one of the best-studied lattices in combinatorics is the Tamari lattice $\mathcal{T}_n$ introduced by Tamari in [118]. Among other things Tamari considered the set T_n of so-called *n-bracketings*, which is the set of all possible ways of inserting n pairs of parentheses into a string of length $n+1$ such that each opening parenthesis has a uniquely associated closing parenthesis, and such that each pair of parentheses contains precisely two factors that are either smaller bracketings or letters. He then defined a hierarchy on T_n induced by the following associativity rule

$$(ab)c \to a(bc), \tag{2.1}$$

where one application of this rule means going up one step in this hierarchy. The *Tamari lattice* $\mathcal{T}_n$ is then the set T_n equipped with the induced partial order. Figure 9 shows the lattices $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$. It was already conjectured in Tamari's thesis [118] that $\mathcal{T}_n$ is indeed a lattice, but it took some more years until this result was proven [56, 62, 120]. A recent reference on the history of Tamari's work along with its impact on different fields of mathematics is [86].

Another property that makes $\mathcal{T}_n$ a frequently recurring combinatorial object is the fact that its cardinality is given by the n-th Catalan number $\mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}$, see [119]. Accordingly, there are many realizations of $\mathcal{T}_n$ as a partial order on Catalan objects. In this chapter, we are mainly interested in the realization of $\mathcal{T}_n$ as a poset on Dyck paths equipped with the so-called rotation order.

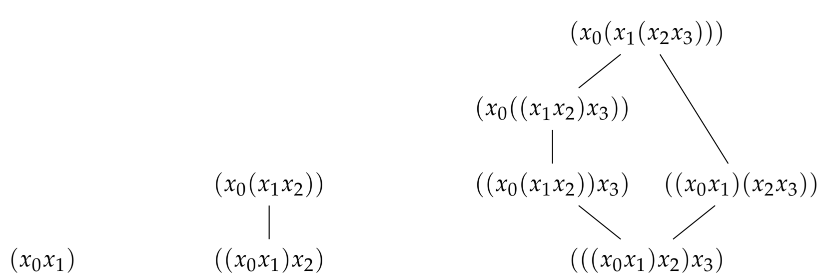


Figure 9. The first three Tamari lattices $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$.

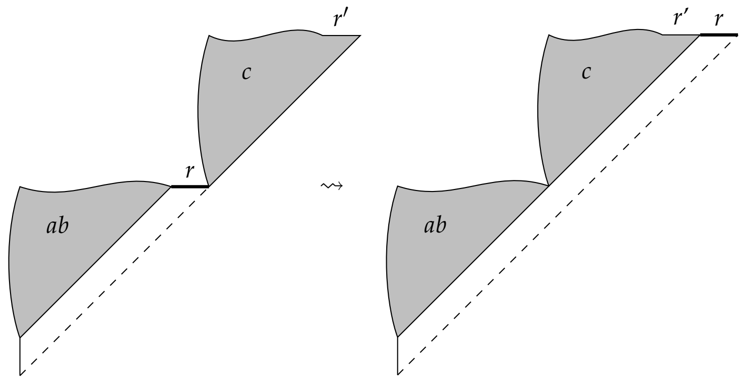


FIGURE 10. The rotation on Dyck paths.

Recall that a *Dyck path of length* $2n$ is a lattice path from $(0,0)$ to (n,n) that consists only of right-steps (i.e. steps of the form $(1,0)$) or up-steps (i.e. steps of the form $(0,1)$), and that never goes below the line $x = y$. We denote the set of Dyck paths of length $2n$ by $\mathcal{D}_n$. Now suppose that $\mathfrak{q} \in \mathcal{D}_n$ is a Dyck path of length $2n$, and let r be a right-step of $\mathfrak{q}$ that is followed by an up-step u. Further let $\mathfrak{q}'$ be the unique subpath of $\mathfrak{q}$ that starts with u, and that is a Dyck path of length $2n'$, where $n' < n$. Let $\varrho_r(\mathfrak{q})$ denote the unique Dyck path in $\mathcal{D}_n$ that is created from $\mathfrak{q}$ by exchanging r and $\mathfrak{q}'$. The *rotation order* is the partial order on $\mathcal{D}_n$ whose cover relations are given by

$$\mathfrak{q}_1 \lessdot_{\mathrm{rot}} \mathfrak{q}_2 \quad \text{if and only if} \quad \mathfrak{q}_2 = \varrho_r(\mathfrak{q}_1) \text{ for some right-step } r \text{ of } \mathfrak{q}_1. \tag{2.2}$$

See Figure 10 for an illustration of the rotation order, and see Figure 11 for an illustration of $\mathcal{T}_4$ as a poset on Dyck paths.

The fact that the poset $(\mathcal{D}_n, \leq_{\mathrm{rot}})$ is indeed isomorphic to $\mathcal{T}_n$ can be seen best by taking a detour through binary trees. By definition, a bracketing $b \in T_n$ can be written in the form $b = (b_1 b_2)$, where b_1 and b_2 are either smaller bracketings or single letters. Thus it is obvious that we can encode each bracketing by a binary tree where b_1 corresponds to the left subtree of the root and b_2 corresponds to the right subtree of the root and such a bracketing corresponds to a leaf if and only if it consists of a single letter. Then, the associativity rule from (2.1) corresponds to the *right rotation* of binary trees illustrated in Figure 12. There is a well-known bijection between binary trees with $n+1$ leaves and Dyck paths of length $2n$, see for instance [8]: we construct a Dyck path by parsing the tree depth-first (i.e. we first visit the left child, then the right child, then the node itself) and for every leaf we visit (except for the first), we add an up-step, and for every inner node we visit, we add a right-step. In view of this bijection, it is immediately clear that the right rotation on binary trees corresponds to the rotation on Dyck paths.

Among the many combinatorial properties of the Tamari lattice, we will now recall two results that have strong consequences for the topology of $\mathcal{T}_n$, and that have motivated the research presented in this chapter: the possible values of the Möbius function of $\mathcal{T}_n$ have been determined by PALLO, who gave an explicit algorithm for their computation.

THEOREM 2.1.1 ([89, Section 5])

For $n > 0$ the Möbius function of $\mathcal{T}_n$ takes values only in $\{-1, 0, 1\}$.

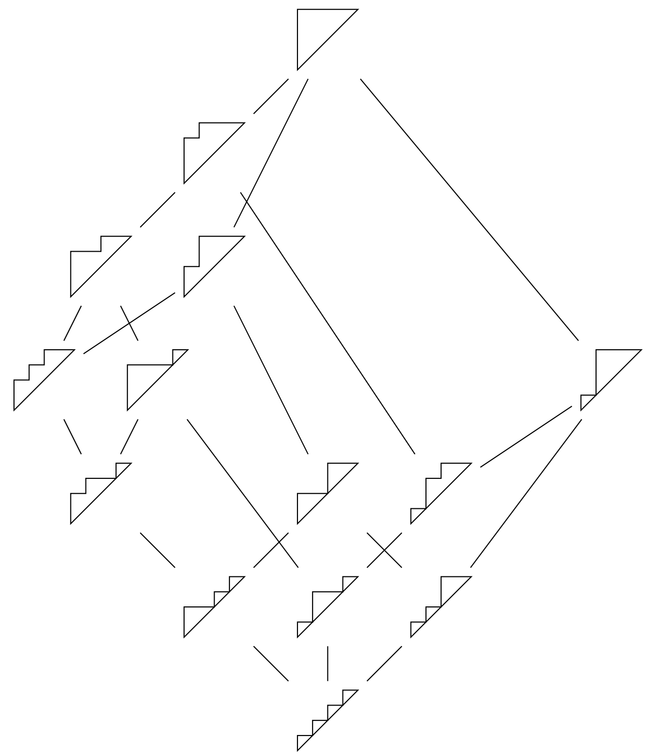


FIGURE 11. The Tamari lattice $\mathcal{T}_4$ as a poset on Dyck paths.

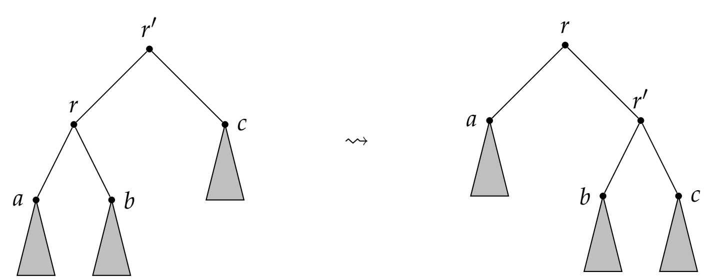


FIGURE 12. The right rotation on binary trees.

This result was later recovered by BJÖRNER and WACHS, who proved that $\mathcal{T}_n$ is EL-shellable and that (with respect to their labeling) there is at most one falling chain in each interval. Moreover, they characterized the spherical and the contractible intervals of $\mathcal{T}_n$.

THEOREM 2.1.2 ([26, Theorem 9.2])

For $n > 0$ the lattice $\mathcal{T}_n$ is EL-shellable.

Besides this, there is a deep connection between the Tamari lattices and the dimension and the graded Frobenius characteristic of the space of diagonal harmonic polynomials in two sets of variables. This was first observed in [60, Fact 2.8.1], and [59, Conjecture 3.1.2], and it was later generalized in [11]. The main object with which we are working in this chapter is a generalization of the Tamari lattice called the m-Tamari lattice. This generalization was first considered in [11, Section 5], and served as a valuable tool for the computation of the graded Frobenius characteristic of the spaces of higher diagonal harmonic polynomials in $m+1$ sets of variables.

In this chapter, we formally define these lattices, and we investigate their topology as well as some structural properties. In particular, we prove that these lattices are EL-shellable, and we compute their Möbius function, see Theorem 2.3.1. Finally, we give a new realization of $\mathcal{T}_n^{(m)}$ in terms of m-tuples of classical Dyck paths, see Theorem 2.4.24, and we define a family of "m-Tamari like" lattices for the dihedral groups, see Section 2.4.4.

2.2. Definition and Examples

We start right away with the definition of the main objects of this chapter.

Definition 2.2.1

For $m,n>0$ we say that an *m-Dyck path of length $(m+1)n$* is a lattice path from $(0,0)$ to (mn,n) that consists only of right-steps and up-steps, and that never goes below the line $y=mx$.

Clearly, the 1-Dyck paths of length $2n$ are precisely the classical Dyck paths of length $2n$. Let $\mathcal{D}_n^{(m)}$ denote the set of all m-Dyck paths of length $(m+1)n$. It is well known that the cardinality of $\mathcal{D}_n^{(m)}$ is given by the *Fuß-Catalan number* $\mathrm{Cat}^{(m)}(n)=\frac{1}{mn+1}\binom{(m+1)n}{n}$, see for instance [45, 69]. Each m-Dyck path $\mathfrak{p}\in\mathcal{D}_n^{(m)}$ comes equipped with two sequences: the *height sequence* $\mathbf{h}_{\mathfrak{p}}=(h_1,h_2,\ldots,h_{mn})$, which satisfies

$$h_1\leq h_2\leq\cdots\leq h_{mn} \tag{2.3}$$

$$\left\lceil\frac{i}{m}\right\rceil\leq h_i\leq n,\quad\text{for all } i\in\{1,2,\ldots,mn\}, \tag{2.4}$$

and the *step sequence* $\mathbf{u}_{\mathfrak{p}}=(u_1,u_2,\ldots,u_n)$, which satisfies

$$u_1\leq u_2\leq\cdots\leq u_n \tag{2.5}$$

$$0\leq u_i\leq m(i-1),\quad\text{for all } i\in\{1,2,\ldots,n\}. \tag{2.6}$$

The interpretation of these sequences is that the i-th entry of the height sequence indicates which height the path has at the coordinate $x=i-1$, and the i-th entry of the step sequence indicates at which x-coordinate the i-th up-step takes place. The next lemma, the proof of which is straightforward and hence omitted, implies how to convert these sequences into each other.

Lemma 2.2.2

Let $\mathfrak{p}\in\mathcal{D}_n^{(m)}$ with step sequence $\mathbf{u}_{\mathfrak{p}}=(u_1,u_2,\ldots,u_n)$ and height sequence $\mathbf{h}_{\mathfrak{p}}=(h_1,h_2,\ldots,h_{mn})$. If we set $h_0=0$, then we have

$$u_{h_i+1}=u_{h_i+2}=\cdots=u_{h_{i+1}}=i,\ \textit{for all } i\in\{0,1,\ldots,mn\}\ \textit{with } h_i<h_{i+1},\quad\textit{and}$$
$$h_i=\max\{j\in\{1,2,\ldots,n\}\mid u_j<i\},\ \textit{for all } i\in\{1,2,\ldots,mn\}.$$

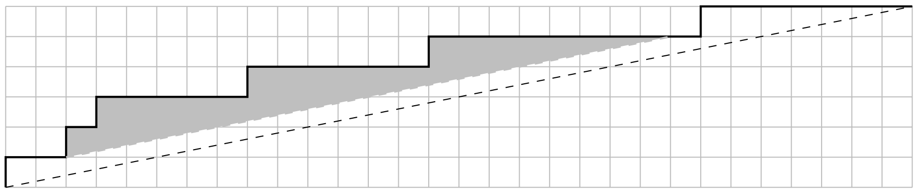


FIGURE 13. A 5-Dyck path of length 36.

EXAMPLE 2.2.3

Figure 13 shows a Dyck path $\mathfrak{p} \in \mathcal{D}_6^{(5)}$. Its height sequence is given by

$$\mathbf{h}_{\mathfrak{p}} = (1,1,2,3,3,3,3,3,4,4,4,4,4,4,5,5,5,5,5,5,5,5,5,6,6,6,6,6,6,6),$$

and its step sequence is given by $\mathbf{u}_{\mathfrak{p}} = (0,2,3,8,14,23)$.

Analogously to classical Dyck paths, we can define a rotation order on $\mathcal{D}_n^{(m)}$, following the lines before (2.2). According to [11, Section 5], this definition can also be rephrased in terms of step sequences. Let $\mathfrak{p} \in \mathcal{D}_n^{(m)}$ have step sequence $\mathbf{u}_{\mathfrak{p}} = (u_1, u_2, \ldots, u_n)$. For $i \in \{1,2,\ldots,n\}$, we call the unique subsequence $(u_i, u_{i+1}, \ldots, u_k)$ of $\mathbf{u}_{\mathfrak{p}}$ satisfying

$$u_j - u_i < m(j-i), \text{ for all } j \in \{i, i+1, \ldots, k\}, \tag{2.7}$$

$$u_{k+1} - u_i \geq m(k+1-i) \quad \text{or} \quad k = n. \tag{2.8}$$

the *primitive subsequence of* $\mathbf{u}_{\mathfrak{p}}$ *at position* i. For $\mathfrak{p}, \mathfrak{p}' \in \mathcal{D}_n^{(m)}$ with $\mathbf{u}_{\mathfrak{p}} = (u_1, u_2, \ldots, u_n)$, we define

(2.9) $\mathfrak{p} \lessdot_{\mathrm{rot}} \mathfrak{p}'$ if and only if

$$\mathbf{u}_{\mathfrak{p}'} = (u_1, u_2, \ldots, u_{i-1}, u_i - 1, u_{i+1} - 1, \ldots, u_k - 1, u_{k+1}, u_{k+2}, \ldots, u_n),$$

for some $i \in \{2,3,\ldots,n\}$ with $u_{i-1} < u_i$, where $(u_i, u_{i+1}, \ldots, u_k)$ is the primitive subsequence of $\mathbf{u}_{\mathfrak{p}}$ at position i. If $\leq_{\mathrm{rot}}$ denotes the reflexive and transitive closure of $\lessdot_{\mathrm{rot}}$, then we call the poset $\left(\mathcal{D}_n^{(m)}, \leq_{\mathrm{rot}}\right)$ the *m-Tamari lattice of parameter n*, and we denote it by $\mathcal{T}_n^{(m)}$. (This name will be justified soon.) Figure 14 shows the poset $\mathcal{T}_3^{(3)}$.

EXAMPLE 2.2.4

Let $\mathfrak{p}$ be the 5-Dyck path of length 36 shown in Figure 13. The primitive subsequence of $\mathbf{u}_{\mathfrak{p}}$ at position 2 is $(2,3,8,14)$, and the highlighted part of $\mathfrak{p}$ indicates that this subsequence is the step sequence of an m-Dyck path of smaller length in its own right. Hence it is immediately clear that the definition of the rotation order in (2.9) coincides with the generalization of the one given in (2.2).

DISCLAIMER 2.2.5

If $m = 0$ or $n = 0$, then the resulting poset is a singleton and hence trivial. Thus in what follows, we will always assume $m, n > 0$.

The next easy lemma states how the step and the height sequences of m-Dyck paths behave with respect to the rotation order. Again, we omit the proof which is straightforward from the definition.

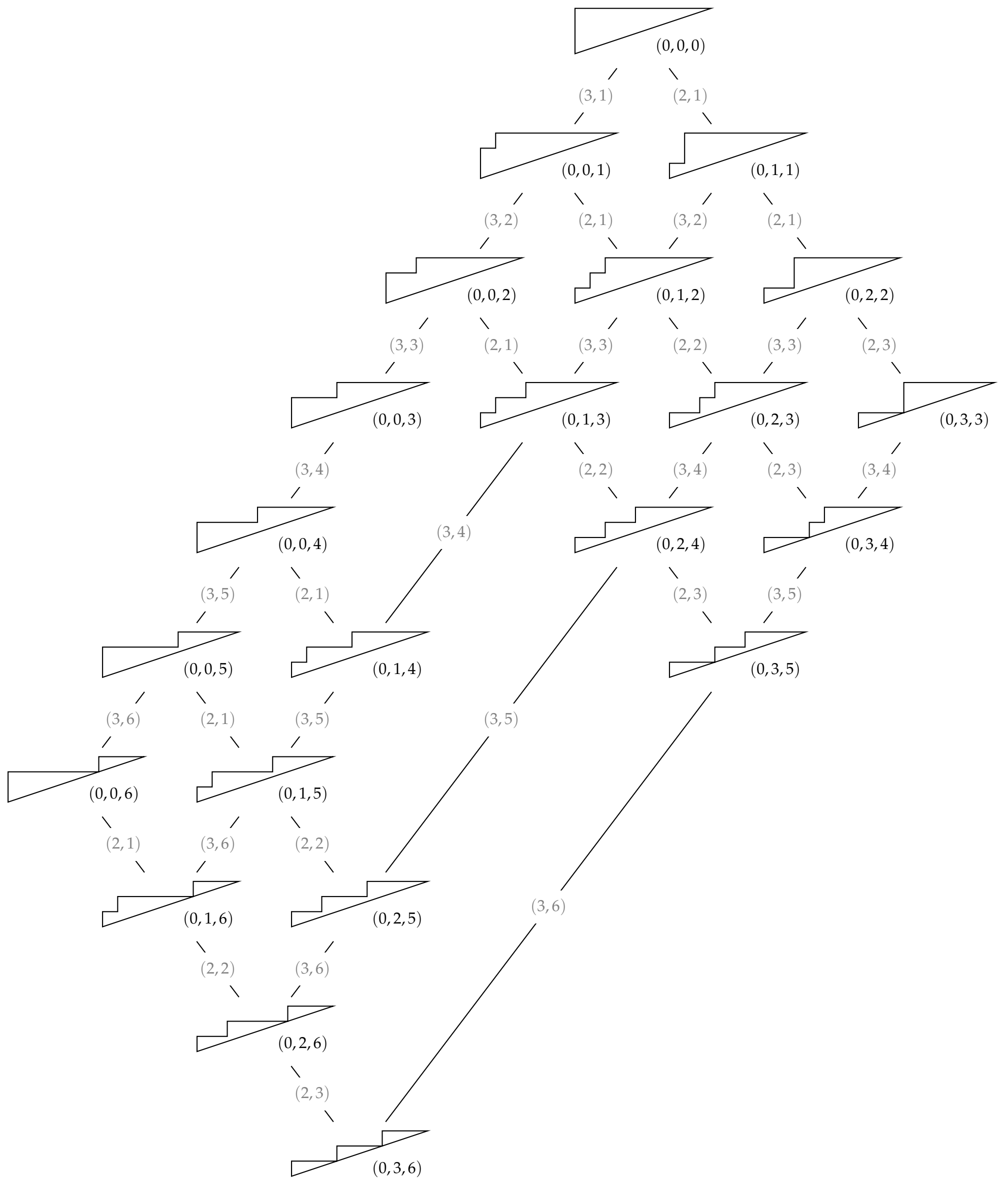


FIGURE 14. The lattice $\mathcal{T}_3^{(3)}$ as a poset on 3-Dyck paths. Each 3-Dyck path is displayed together with its step sequence, and the edges are labeled by the edge-labeling defined in (2.10).

LEMMA 2.2.6

Let $m, n > 0$, and let $\mathfrak{p}, \mathfrak{p}' \in \mathcal{D}_n^{(m)}$ with $\mathfrak{p} \leq_{\mathrm{rot}} \mathfrak{p}'$. If $\mathbf{u}_{\mathfrak{p}} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{u}_{\mathfrak{p}'} = (u'_1, u'_2, \ldots, u'_n)$ denote the step sequences of $\mathfrak{p}$ and $\mathfrak{p}'$, respectively, then we have $u_i \geq u'_i$ for all $i \in \{1, 2, \ldots, n\}$. Moreover, if $\mathbf{h}_{\mathfrak{p}} = (h_1, h_2, \ldots, h_{mn})$ and $\mathbf{h}_{\mathfrak{p}'} = (h'_1, h'_2, \ldots, h'_{mn})$ denote the height sequences of $\mathfrak{p}$ and $\mathfrak{p}'$, respectively, then we have $h_i \leq h'_i$ for all $i \in \{1, 2, \ldots, mn\}$.

A crucial observation about the poset $\mathcal{T}_n^{(m)}$ is that it can be embedded as an interval in $\mathcal{T}_{mn}$, which implies that it is indeed a lattice. More precisely, every m-Dyck path of length $(m+1)n$ can be converted into a classical Dyck path of length $2mn$ by replacing each up-step by m consecutive up-steps. In view of this construction, it follows immediately that a cover relation between two m-Dyck paths corresponds exactly to a cover relation between the associated "blown-up" Dyck paths. Hence we have the following result.

Proposition 2.2.7 ([30, Proposition 4])

For $n, m > 0$, the poset $\mathcal{T}_n^{(m)}$ is isomorphic to the interval $[\mathfrak{q}, \mathfrak{q}']$ in $\mathcal{T}_{mn}$, where $\mathfrak{q}$ is the Dyck path whose step sequence $\mathbf{u}_{\mathfrak{q}} = (u_1, u_2, \ldots, u_{mn})$ is given by $u_{in+1} = u_{in+2} = \cdots = u_{in+n} = i$, for $i \in \{0, 1, \ldots, m-1\}$, and where $\mathfrak{q}'$ is the Dyck path whose step sequence is $\mathbf{u}_{\mathfrak{q}'} = (0, 0, \ldots, 0)$.

Remark 2.2.8

In [3], a further generalization of Dyck paths was considered, namely so-called *(a, b)-Dyck paths*, where a and b are coprime. These paths are lattice paths from $(0, 0)$ to (b, a) that consist only of right-steps and up-steps and never go below the line $y = \frac{a}{b}x$. If we denote the set of all (a, b)-Dyck paths by $\mathcal{D}_{a,b}$, then it follows from [20] that

$$\left|\mathcal{D}_{a,b}\right| = \frac{1}{a+b}\binom{a+b}{a, b} = \frac{(a+b-1)!}{a!\, b!},$$

see also [3, Theorem 3.1]. Again we can define a rotation order on these paths analogous to before, and the resulting poset $\mathcal{T}_{a,b} = \left(\mathcal{D}_{a,b}, \leq_{\mathrm{rot}}\right)$ is again an interval in $\mathcal{T}_b$. See Figure 15 for an illustration of the poset $\mathcal{T}_{4,7}$. In the case $a = n$ and $b = n + 1$, we obtain precisely the Dyck paths of length $2n$, and in the case $a = n$ and $b = mn + 1$, we obtain precisely the m-Dyck paths of length $(m+1)n$.

2.3. Topological Properties of $\mathcal{T}_n^{(m)}$

In view of Proposition 2.2.7, the results in Theorems 2.1.1 and 2.1.2 can be generalized immediately to $\mathcal{T}_n^{(m)}$ for $m > 1$. However, we will reprove these results independently, which at the same time provides a uniform proof for all m-Tamari lattices. The main result of this section is the following.

Theorem 2.3.1

For $m, n > 0$ the m-Tamari lattice of parameter n is EL-shellable. Moreover, the Möbius function of $\mathcal{T}_n^{(m)}$ takes values only in $\{-1, 0, 1\}$.

First of all we need to find suitable edge-labeling which embodies the characteristic properties of the cover relations in $\mathcal{T}_n^{(m)}$. By definition, a cover relation $\mathfrak{p} \lessdot_{\mathrm{rot}} \mathfrak{p}'$ is uniquely determined by some up-step of $\mathfrak{p}$. In terms of step sequences this means the following: let $\mathfrak{p}, \mathfrak{p} \in \mathcal{D}_n^{(m)}$ with $\mathfrak{p} \lessdot_{\mathrm{rot}} \mathfrak{p}'$ where the step sequences are $\mathbf{u}_{\mathfrak{p}} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{u}_{\mathfrak{p}'} = (u_1', u_2', \ldots, u_n')$, respectively. Then, there exists a minimal index $i \in \{2, 3, \ldots, n\}$ such that $u_i' = u_i - 1$, and we will assign this index together with the corresponding entry in $\mathbf{u}_{\mathfrak{p}}$ to the edge between $\mathfrak{p}$ and $\mathfrak{p}'$ in $\mathcal{T}_n^{(m)}$. More precisely, we consider the following edge-labeling of

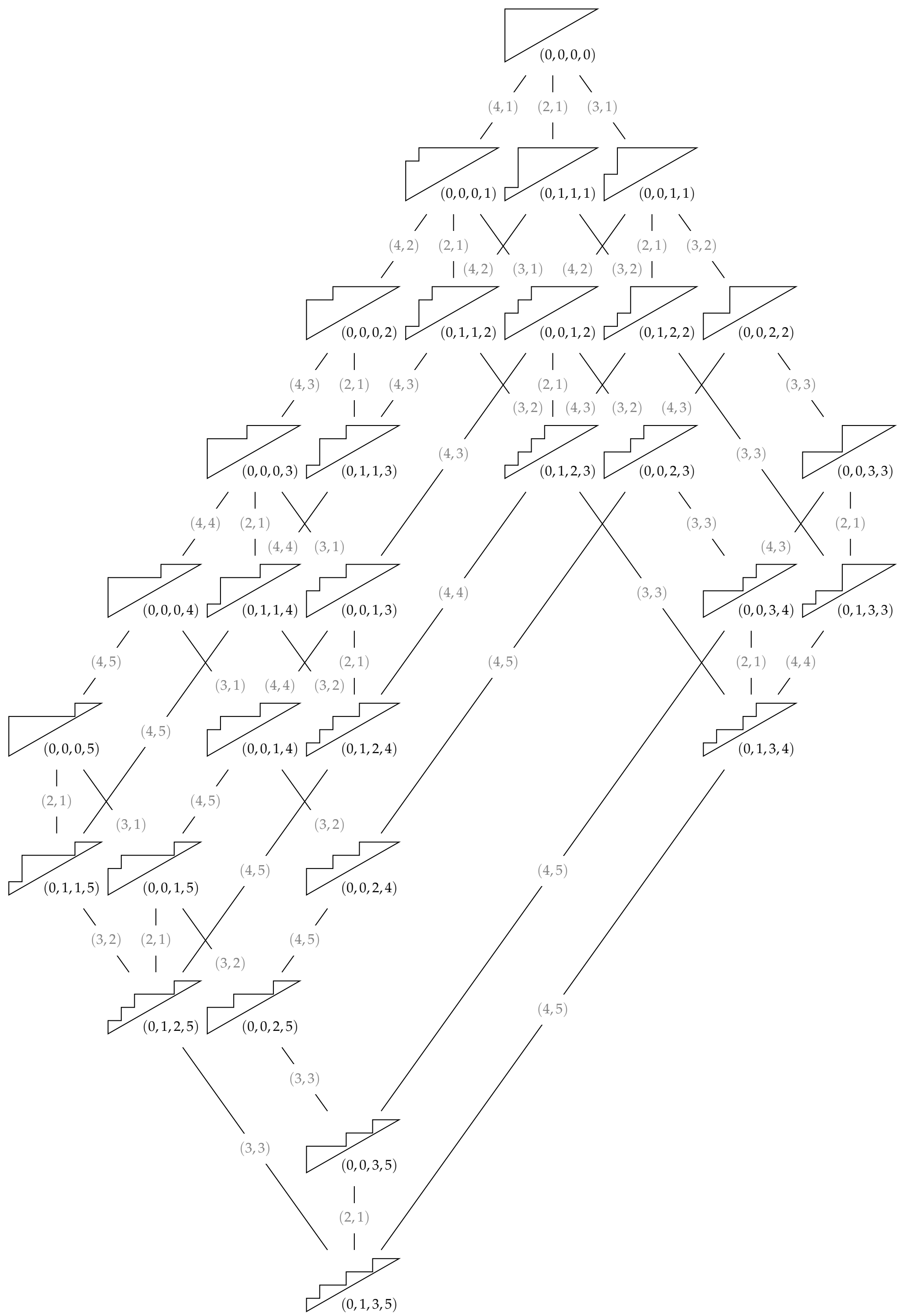


FIGURE 15. The lattice $\mathcal{T}_{4,7}$.

$\mathcal{T}_n^{(m)}$:

$$\lambda : \mathcal{E}\big(\mathcal{T}_n^{(m)}\big) \to \mathbb{N}\times\mathbb{N}, \quad (\mathfrak{p},\mathfrak{p}') \mapsto (i,u_i), \tag{2.10}$$

where $\mathbf{u}_{\mathfrak{p}} = (u_1,u_2,\ldots,u_n)$ and $\mathbf{u}_{\mathfrak{p}'} = (u_1',u_2',\ldots,u_n')$ and $i = \min\{j \mid u_j \neq u_j'\}$. See Figure 14 for an illustration of this labeling. Then, we have the following result, which immediately implies Theorem 2.3.1.

Theorem 2.3.2

For $m,n > 0$ the labeling from (2.10) is an EL-labeling for $\mathcal{T}_n^{(m)}$ with respect to the following total order on $\mathbb{N}\times\mathbb{N}$:

$$(i,u_i) \preceq (j,u_j) \quad \text{if and only if} \quad i<j \quad \text{or} \quad i=j \text{ and } u_i \geq u_j.$$

Moreover, there is at most one falling chain in every interval of $\mathcal{T}_n^{(m)}$ with respect to this labeling.

Proof. Let $m,n > 0$, and let $\mathfrak{p},\mathfrak{p}' \in \mathcal{D}_n^{(m)}$ satisfy $\mathfrak{p} <_{\mathrm{rot}} \mathfrak{p}'$. We need to show that there is a unique rising maximal chain in the interval $[\mathfrak{p},\mathfrak{p}']$ and that this chain is lexicographically first among all maximal chains in this interval. Let $\mathbf{u}_{\mathfrak{p}} = (u_1,u_2,\ldots,u_n)$, and let $\mathbf{u}_{\mathfrak{p}'} = (u_1',u_2',\ldots,u_n')$, and consider the set of differences $D = \{j \in \{1,2,\ldots,n\} \mid u_j \neq u_j'\}$. We write the elements of D in increasing order, i.e. $D = \{j_1,j_2,\ldots,j_s\}$, where $j_1 < j_2 < \cdots < j_s$.

Let $\mathfrak{r}^{(0)} = \mathfrak{p}$, and construct $\mathfrak{r}^{(i+1)}$ from $\mathfrak{r}^{(i)}$ by decreasing the entries of the primitive subsequence of $\mathbf{u}_{\mathfrak{r}^{(i)}}$ at position j_k by one, where j_k is the smallest element in D such that the j_k-th entry of $\mathbf{u}_{\mathfrak{r}^{(i)}}$ is larger than the j_k-th entry of $\mathbf{u}_{\mathfrak{p}'}$. By the minimality of j_k, it is always guaranteed that $\mathfrak{r}^{(i+1)} \leq_{\mathrm{rot}} \mathfrak{p}'$. Since $\mathcal{T}_n^{(m)}$ is finite we eventually reach an index t such that $\mathfrak{r}^{(t)} = \mathfrak{p}'$. By construction it follows immediately that the chain

$$\mathfrak{p} = \mathfrak{r}^{(0)} \lessdot_{\mathrm{rot}} \mathfrak{r}^{(1)} \lessdot_{\mathrm{rot}} \cdots \lessdot_{\mathrm{rot}} \mathfrak{r}^{(t)} = \mathfrak{p}' \tag{2.11}$$

is rising. Moreover, it follows by construction that $\lambda(\mathfrak{p},\mathfrak{r}^{(1)}) = (j_1,u_{j_1})$. Now let $\mathfrak{r} \in \mathcal{D}_n^{(m)}$ with $\mathfrak{p} \lessdot_{\mathrm{rot}} \mathfrak{r} \leq_{\mathrm{rot}} \mathfrak{p}'$ where $\mathfrak{r} \neq \mathfrak{r}^{(1)}$. It follows from the uniqueness of primitive subsequences that $\lambda(\mathfrak{p},\mathfrak{r}) = (j_k,u_{j_k})$ for $k > 1$, which immediately implies that the chain in (2.11) is the lexicographically first maximal chain in $[\mathfrak{p},\mathfrak{p}']$. Moreover, the j_1-st value of the step sequence of $\mathfrak{r}$ is u_{j_1} which is strictly larger than u_{j_1}'. Hence in every maximal chain from $\mathfrak{r}$ to $\mathfrak{p}'$, there exists an edge labeled by (j_1,u), which implies that the chain in (2.11) is the unique rising maximal chain in the interval $[\mathfrak{p},\mathfrak{p}']$. Hence λ is an EL-labeling for $\mathcal{T}_n^{(m)}$.

Now we want to count the falling chains in $[\mathfrak{p},\mathfrak{p}']$. Consider the set $D' = \{j \in \{1,2,\ldots,n\} \mid u_j \neq u_j' \text{ and } u_j \geq u_{j-1}+m\}$. This time we write the elements of D' in decreasing order, namely $D' = \{j_1',j_2',\ldots,j_s'\}$ with $j_1' > j_2' > \cdots > j_s'$. (This s is not necessarily the same as in the previous paragraph.) The interpretation of D' is the following: if $j \in D'$, then the step sequences of $\mathfrak{p}$ and $\mathfrak{p}'$ differ in position j, and u_j is not contained in the primitive subsequence of $\mathbf{u}_{\mathfrak{p}}$ at some position $< j$. Let $\bar{\mathfrak{r}}^{(0)} = \mathfrak{p}$, and for $i \in \{1,2,\ldots,s\}$, construct $\bar{\mathfrak{r}}^{(i)}$ from $\bar{\mathfrak{r}}^{(i-1)}$ by decreasing the entries of the primitive subsequence of $\bar{\mathfrak{r}}^{(i)}$ at position j_i' by one. By construction, we have $\bar{\mathfrak{r}}^{(i-1)} \lessdot_{\mathrm{rot}} \bar{\mathfrak{r}}^{(i)}$ for $i \in \{1,\ldots,s\}$, and it follows that $\lambda\big(\bar{\mathfrak{r}}^{(i-1)},\bar{\mathfrak{r}}^{(i)}\big) = (j_i',u_{j_i'})$. Hence the chain

$$\mathfrak{p} = \bar{\mathfrak{r}}^{(0)} \lessdot_{\mathrm{rot}} \bar{\mathfrak{r}}^{(1)} \lessdot_{\mathrm{rot}} \cdots \lessdot_{\mathrm{rot}} \bar{\mathfrak{r}}^{(s)} \tag{2.12}$$

is falling. Moreover, suppose that there are three elements $\mathfrak{r}_1,\mathfrak{r}_2,\mathfrak{r}_3 \in \mathcal{D}_n^{(m)}$ with $\mathfrak{r}_1 \lessdot_{\mathrm{rot}} \mathfrak{r}_2 \lessdot_{\mathrm{rot}} \mathfrak{r}_3$ as well as $\lambda(\mathfrak{r}_1,\mathfrak{r}_2) = (k,u)$, and $\lambda(\mathfrak{r}_2,\mathfrak{r}_3) = (k,u')$. Then, it follows by construction that $u > u'$, hence such a chain cannot be falling. This implies that the chain in (2.12) is the only possible falling chain in $[\mathfrak{p},\mathfrak{p}']$, and this chain is maximal if and only if $\bar{\mathfrak{r}}^{(s)} = \mathfrak{p}'$. Hence the proof is complete. □

PROOF OF THEOREM 2.3.1. This follows by definition from Theorem 2.3.2. □

In view of the proof of Theorem 2.3.2, we can immediately compute the length of $\mathcal{T}_n^{(m)}$.

COROLLARY 2.3.3

For $m,n>0$ we have $\ell\big(\mathcal{T}_n^{(m)}\big) = m\binom{n}{2}$.

PROOF. Let $\mathfrak{o}$ and $\mathfrak{1}$ denote the least and the greatest element of $\mathcal{T}_n^{(m)}$. By definition, we have $\mathbf{u}_{\mathfrak{o}} = (0,m,2m,\ldots,(n-1)m)$, and $\mathbf{u}_{\mathfrak{1}} = (0,0,\ldots,0)$. The maximal rising chain from $\mathfrak{o}$ to $\mathfrak{1}$ has length $m\binom{n}{2}$, since in view of (2.11), we first decrease the second entry of $\mathbf{u}_{\mathfrak{o}}$ until it is equal to 0, then the third until it is equal to 0, and so on. Summing this up yields $\sum_{i=2}^{n} m(i-1) = m\binom{n}{2}$. Now, Lemma 1.1.6 implies that the lexicographic first maximal chain in an EL-shellable poset has maximal length, which implies the result. □

REMARK 2.3.4

We can analogously to (2.10) define an edge-labeling for the rational Tamari lattices $\mathcal{T}_{a,b}$, see Remark 2.2.8, and the proof of Theorem 2.3.2 can be carried over almost verbatim.

2.3.1. The Möbius Function.

In this section, we use the EL-labeling (2.10) to compute the Möbius function of $\mathcal{T}_n^{(m)}$. The next result follows immediately from Theorem 2.3.2, but in view of Proposition 2.2.7 and Theorem 2.1.1 is not new. However, using Theorem 2.3.2, we can prove this result simultaneously for all $m,n>0$.

COROLLARY 2.3.5

For $m,n>0$ the Möbius function of $\mathcal{T}_n^{(m)}$ takes values only in $\{-1,0,1\}$. Hence every interval in $\mathcal{T}_n^{(m)}$ is either spherical or contractible.

PROOF. It follows from Theorem 2.3.2 that in every interval of $\mathcal{T}_n^{(m)}$ there is at most one falling maximal chain with respect to the edge-labeling (2.10). Hence Proposition 1.1.14 implies that the Möbius function of $\mathcal{T}_n^{(m)}$ takes only values in $\{-1,0,1\}$. Moreover, Theorem 1.1.16 implies that the order complex of every interval of $\mathcal{T}_n^{(m)}$ has at most one non-vanishing reduced Betti number, which implies that it is either spherical or contractible. □

In the remaining part of this section, we characterize the spherical intervals of $\mathcal{T}_n^{(m)}$, and we compute the number of spherical intervals of the form $[\mathfrak{o},\mathfrak{p}]$ or $[\mathfrak{p},\mathfrak{1}]$. For that, let $\mathfrak{p}\in\mathcal{D}_n^{(m)}$, and let $\mathbf{u}_{\mathfrak{p}} = (u_1,u_2,\ldots,u_n)$ be its step sequence. Define $D_{\mathfrak{p}} = \big\{i\in\{1,2,\ldots,n\} \mid u_i > u_{i-1}\big\}$, and

$$\mathrm{ps}_{\mathfrak{p}}(j) = \big|\{i\in D_{\mathfrak{p}} \mid i<j \text{ and } u_j - 1 - u_i < m(j-i) \text{ and } u_k - u_i < m(k-i) \text{ for all } i<k<j\}\big| \tag{2.13}$$

for all $j \in D_{\mathfrak{p}}$. Fix $j \in D_{\mathfrak{p}}$. If $\mathfrak{p}_j$ denotes the upper cover of $\mathfrak{p}$ in $\mathcal{T}_n^{(m)}$ whose step sequence is given by decreasing the entries of the primitive subsequence of $\mathbf{u}_{\mathfrak{p}}$ at position j by one, then $\mathrm{ps}_{\mathfrak{p}}(j)$ counts the primitive subsequences of $\mathbf{u}_{\mathfrak{p}_j}$ at position $i \in D_{\mathfrak{p}}$ that contain the j-th entry of $\mathbf{u}_{\mathfrak{p}_j}$.

Example 2.3.6

Let us consider the 5-Dyck path $\mathfrak{p}$ of length 36 from Figure 13 again, which is given by the step sequence $\mathbf{u}_{\mathfrak{p}} = (0,2,3,8,14,23)$. Then $D_{\mathfrak{p}} = \{2,3,4,5\}$, and we have

$$\mathrm{ps}_{\mathfrak{p}}(2) = 0, \quad \mathrm{ps}_{\mathfrak{p}}(3) = 1, \quad \mathrm{ps}_{\mathfrak{p}}(4) = 2, \quad \mathrm{ps}_{\mathfrak{p}}(5) = 1.$$

If we take for instance $j = 5$, then the upper cover $\mathfrak{p}_5$ of $\mathfrak{p}$ that is constructed from decreasing the primitive subsequence of $\mathbf{u}_{\mathfrak{p}}$ at position 5 is given by $\mathbf{u}_{\mathfrak{p}_5} = (0,2,3,8,13,23)$. If $\mathbf{u}_{\mathfrak{p}_5}(j)$ denotes the primitive subsequence of $\mathbf{u}_{\mathfrak{p}_5}$ at position j, then we have

$$\mathbf{u}_{\mathfrak{p}_5}(2) = (2,3,8,13), \quad \mathbf{u}_{\mathfrak{p}_5}(3) = (3), \quad \mathbf{u}_{\mathfrak{p}_5}(4) = (8).$$

The fifth entry of $\mathbf{u}_{\mathfrak{p}_5}$, which is 13, is only contained in one of those subsequences, and it follows that $\mathrm{ps}_{\mathfrak{p}}(5) = 1$.

Theorem 2.3.7

Let $m,n > 0$, and let $\mathfrak{p},\mathfrak{p}' \in \mathcal{D}_n^{(m)}$ with $\mathfrak{p} <_{\mathrm{rot}} \mathfrak{p}'$ as well as $\mathbf{u}_{\mathfrak{p}} = (u_1,u_2,\ldots,u_n)$ and $\mathbf{u}_{\mathfrak{p}'} = (u_1',u_2',\ldots,u_n')$. Let $D = \{j \in \{1,2,\ldots,n\} \mid u_j \neq u_j' \text{ and } u_j > u_{j-1}\}$. Then the open interval $(\mathfrak{p},\mathfrak{p}')$ has the homotopy type of a $(|D|-2)$-sphere if and only if

$$u_j - 1 - u_{j-1} < m \quad \text{implies} \quad u_j' - u_{j-1}' < m, \quad \text{and} \tag{2.14}$$

$$u_j' = u_j - 1 - \mathrm{ps}_{\mathfrak{p}}(j), \tag{2.15}$$

for all $j \in D$. Otherwise $(\mathfrak{p},\mathfrak{p}')$ is contractible.

Proof. We need to show that there exists a falling maximal chain in the interval $[\mathfrak{p},\mathfrak{p}']$ if and only if (2.14) and (2.15) are satisfied. For that, we write the elements of D again in decreasing order, namely $D = \{j_1,j_2,\ldots,j_s\}$ with $j_1 > j_2 > \cdots > j_s$. Let $\mathfrak{r}^{(0)} = \mathfrak{p}$, and for $i \in \{1,2,\ldots,s\}$ construct $\mathfrak{r}^{(i)}$ from $\mathfrak{r}^{(i-1)}$ by decreasing the entries of the primitive subsequence of $\mathbf{u}_{\mathfrak{r}^{(i-1)}}$ at position j_i by one. By construction, we have $\mathfrak{r}^{(i-1)} \lessdot_{\mathrm{rot}} \mathfrak{r}^{(i)}$ and $\lambda\big(\mathfrak{r}^{(i-1)},\mathfrak{r}^{(i)}\big) = (j_i,u_{j_i})$. Hence the chain $\mathfrak{r}^{(0)} \lessdot_{\mathrm{rot}} \mathfrak{r}^{(1)} \lessdot_{\mathrm{rot}} \cdots \lessdot_{\mathrm{rot}} \mathfrak{r}^{(s)}$ is falling, and we have $\mathrm{ps}_{\mathfrak{r}^{(i)}}(j_k) = \mathrm{ps}_{\mathfrak{p}}(j_k)$ if $k < i$.

First we show that (2.14) is equivalent to $\mathfrak{r}^{(i)} \leq_{\mathrm{rot}} \mathfrak{p}'$ for all $i \in \{1,2,\ldots,s\}$, and we proceed by contradiction. Suppose that there exists some $k \in \{1,2,\ldots,s\}$ such that $u_{j_k} - 1 - u_{j_{k-1}} < m$ and $u_{j_k}' - u_{j_k-1}' \geq m$, and k is minimal with this property. (This means that j_k is the maximal index with this property.) Since $\mathfrak{p} <_{\mathrm{rot}} \mathfrak{p}'$ and $u_{j_k} \neq u_{j_k}'$ Lemma 2.2.6 implies $u_{j_k} > u_{j_k}'$. If we assume $u_{j_k-1} = u_{j_k-1}'$, then our hypothesis implies $u_{j_k} - 1 < u_{j_k}' < u_{j_k}$, which is a contradiction. Hence again with Lemma 2.2.6 it follows that $u_{j_k-1} > u_{j_k-1}'$. Now let $\bar{\mathfrak{r}} \in \mathcal{D}_n^{(m)}$ be the element whose step sequence $\mathbf{u}_{\bar{\mathfrak{r}}} = (\bar{u}_1,\bar{u}_2,\ldots,\bar{u}_n)$ is obtained from $\mathbf{u}_{\mathfrak{r}^{(k)}}$ by successively decreasing the primitive subsequence at position j_k until $\bar{u}_{j_k} = u_{j_k}'$. Clearly we have $\bar{u}_{j_k-1} > u_{j_k-1}'$. However, since u_{j_k} is contained in the primitive subsequence of $\mathbf{u}_{\mathfrak{p}}$ at position $j_k - 1$, it follows that $\bar{u}_{j_k}$ is contained in the primitive subsequence of $\mathbf{u}_{\bar{\mathfrak{r}}}$ at position $j_k - 1$ as well, which implies that $\bar{\mathfrak{r}} \not\leq_{\mathrm{rot}} \mathfrak{p}'$. Now let $\tilde{\mathfrak{r}} \in \mathcal{D}_n^{(m)}$ be the element whose step sequence $\mathbf{u}_{\bar{\mathfrak{r}}} = (\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n)$ is obtained from $\mathbf{u}_{\mathfrak{r}^{(k)}}$ by successively decreasing the primitive subsequence at position $j_k - 1$ until $\bar{u}_{j_k-1} = u'_{j_k-1}$. By assumption it follows that

$$\bar{u}_{j_k} = u_{j_k} - u_{j_k-1} + u'_{j_k-1} < m + 1 + u'_{j_k-1} \leq u'_{j_k} + 1,$$

which implies with Lemma 2.2.6 that $\bar{\mathfrak{r}} \not\leq_{\mathrm{rot}} \mathfrak{p}'$. Hence $\mathfrak{r}^{(k)} \not\leq_{\mathrm{rot}} \mathfrak{p}'$.

The converse implication is straightforward since (2.14) implies that whenever in the construction of $\mathfrak{r}^{(i)}$ from $\mathfrak{r}^{(i-1)}$ some entry $u_k^{(i)}$ of $\mathbf{u}_{\mathfrak{r}^{(i)}} = \left(u_1^{(i)}, u_2^{(i)}, \ldots, u_n^{(i)}\right)$ is decreased for $k > j_i$, then the k-th entry of $\mathbf{u}_{\mathfrak{r}^{(i+1)}}$ belongs to the primitive subsequence of $\mathbf{u}_{\mathfrak{r}^{(i+1)}}$, and the k-th entry of $\mathbf{u}_{\mathfrak{p}'}$ belongs to the primitive subsequence of $\mathbf{u}_{\mathfrak{p}'}$ at position j_i. Hence $\mathfrak{r}^{(i)} \leq_{\mathrm{rot}} \mathfrak{p}'$ for all $i \in \{1, 2, \ldots, s\}$.

Now we show that (2.15) is equivalent to $\mathfrak{r}^{(s)} = \mathfrak{p}'$. Fix $i \in \{1, 2, \ldots, s\}$, and let $\mathbf{u}_{\mathfrak{r}^{(i)}} = (u_1^{(i)}, u_2^{(i)}, \ldots, u_n^{(i)})$. The number $\mathrm{ps}_{\mathfrak{p}}(j_i)$ corresponds to the number of primitive subsequences of $\mathbf{u}_{\mathfrak{r}^{(i)}}$ at some position $k \in D$ with $k < j_i$ that contain $u_{j_i}^{(i)}$. Hence along the chain $\mathfrak{r}^{(1)} \lessdot_{\mathrm{rot}} \mathfrak{r}^{(2)} \lessdot_{\mathrm{rot}} \cdots \lessdot_{\mathrm{rot}} \mathfrak{r}^{(s)}$, the entry $u_{j_i}^{(i)}$ is decreased exactly $\mathrm{ps}_{\mathfrak{p}}(j_i)$-times. By construction, we have $u_{j_i}^{(i)} = u_{j_i} - 1$, which implies $u'_{j_i} = u_{j_i} - 1 - \mathrm{ps}_{\mathfrak{p}}(j_i)$ as desired.

Hence the chain $\mathfrak{r}^{(0)} \lessdot_{\mathrm{rot}} \mathfrak{r}^{(1)} \lessdot_{\mathrm{rot}} \cdots \lessdot_{\mathrm{rot}} \mathfrak{r}^{(s)}$ is a maximal falling chain in $[\mathfrak{p}, \mathfrak{p}']$ if and only if (2.14) and (2.15) are satisfied. $\square$

EXAMPLE 2.3.8

If we continue with Example 2.3.6, then we find that the path $\mathfrak{p}' \in \mathcal{D}_6^{(5)}$ given by the step sequence $\mathbf{u}_{\mathfrak{p}'} = (0, 1, 1, 5, 12, 23)$ has $\mathfrak{p} \leq_{\mathrm{rot}} \mathfrak{p}'$, and satisfies both (2.14) and (2.15). Figure 16 shows the interval $[\mathfrak{p}, \mathfrak{p}']$ in $\mathcal{T}_6^{(5)}$, and we can check that $\mu_{\mathcal{T}_6^{(5)}}(\mathfrak{p}, \mathfrak{p}') = 1$. The unique falling maximal chain in that interval is

$$\begin{aligned}(0,2,3,8,14,23) \lessdot_{\mathrm{rot}} (0,2,3,8,13,23) \lessdot_{\mathrm{rot}} (0,2,3,7,13,23)&\\ \lessdot_{\mathrm{rot}} (0,2,2,6,13,23) \lessdot_{\mathrm{rot}} (0,1,1,5,12,23)&.\end{aligned}$$

The following corollaries are immediate.

COROLLARY 2.3.9

Let $m, n > 0$, and let $\mathfrak{p}, \mathfrak{p}' \in \mathcal{D}_n^{(m)}$ with $\mathfrak{p} <_{\mathrm{rot}} \mathfrak{p}'$ as well as $\mathbf{u}_{\mathfrak{p}} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{u}_{\mathfrak{p}'} = (u'_1, u'_2, \ldots, u'_n)$. Let $D = \{j \in \{1, 2, \ldots, n\} \mid u_j \neq u'_j \text{ and } u_j > u_{j-1}\}$. Then,

$$\mu_{\mathcal{T}_n^{(m)}}(\mathfrak{p}, \mathfrak{p}') = \begin{cases} (-1)^{|D|}, & \text{if (2.14) and (2.15) hold,} \\ 0, & \text{otherwise.} \end{cases}$$

PROOF. This follows from Proposition 1.1.12 and Theorem 2.3.7. $\square$

In what follows, we denote the least element of $\mathcal{T}_n^{(m)}$ by $\mathfrak{o}$, and the greatest element of $\mathcal{T}_n^{(m)}$ by $\mathfrak{1}$.

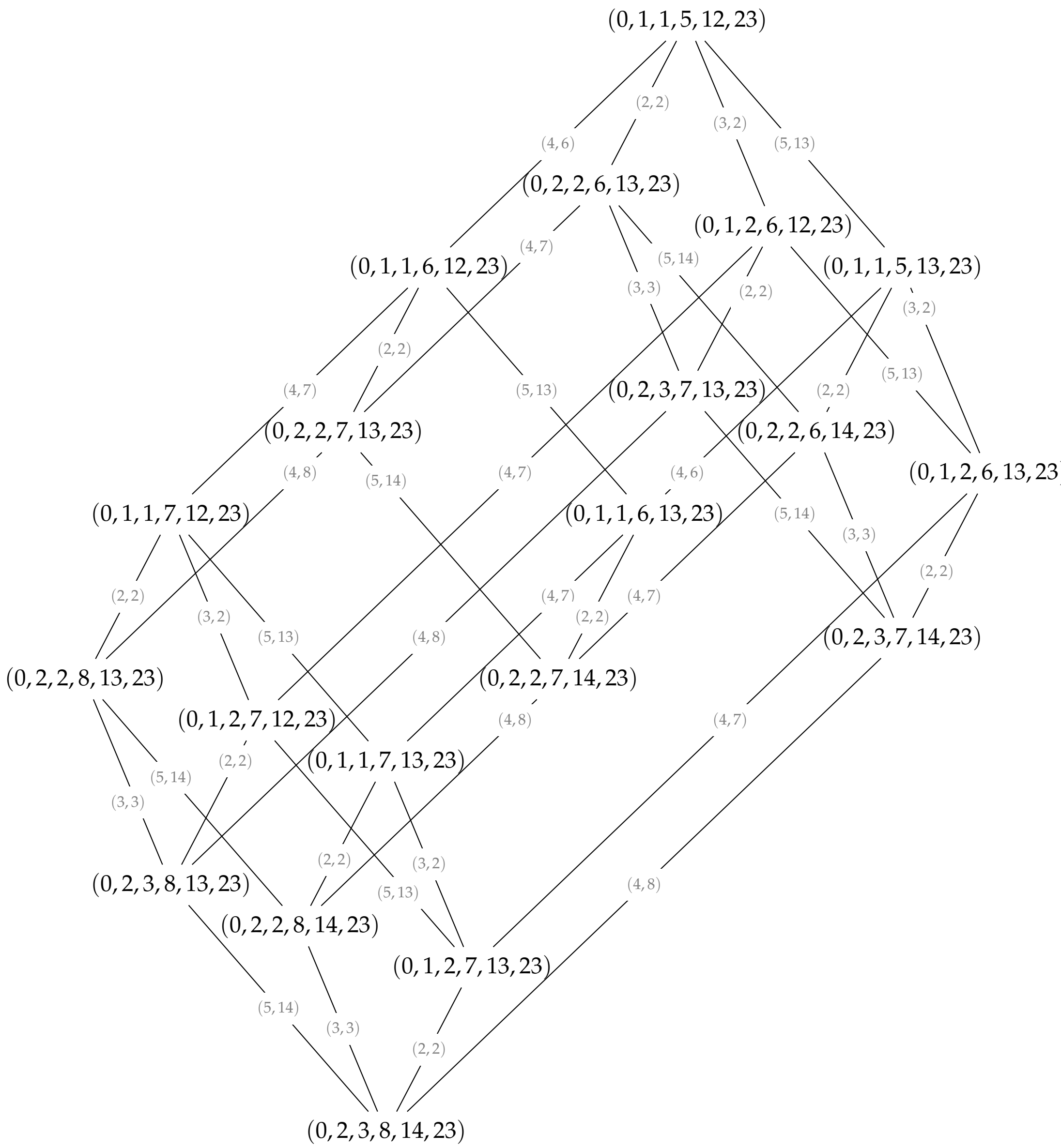


FIGURE 16. A spherical interval in $\mathcal{T}_6^{(5)}$.

COROLLARY 2.3.10

Let $m, n > 0$, and let $\mathfrak{p} \in \mathcal{D}_n^{(m)}$ with $\mathbf{u}_\mathfrak{p} = (u_1, u_2, \ldots, u_n)$. Define $D = \{j \in \{1, 2, \ldots, n\} \mid u_j \neq (j-1)m\}$, and $D_j = \{i \in D \mid i < j\}$. Then,

$$\mu_{\mathcal{T}_n^{(m)}}(\mathfrak{o}, \mathfrak{p}) = \begin{cases} (-1)^{|D|}, & \text{if } u_j = (j-1)m - 1 - |D_j| \text{ for all } j \in D, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. By definition, we have $\mathbf{u}_{\mathfrak{o}} = (0, m, 2m, \ldots, (n-1)m)$. Hence the premise in (2.14) is $(j-1)m - 1 - (j-2)m = m - 1 < m$, and this is always true. Moreover, we have $\mathrm{ps}_{\mathfrak{o}}(j) = \bigl|\{i \in D \mid i < j\}\bigr| = |D_j|$ for all $j \in D$. If $\mathfrak{p}$ satisfies (2.15), then $u_j = (j-1)m - 1 - |D_j|$. It remains to show that under this assumption the conclusion of (2.14) is true. We distinguish two cases:

(i) Let $j - 1 \in D$. We have

$$u_j - u_{j-1} = (j-1)m - 1 - |D_j| - (j-2)m + 1 + |D_{j-1}| = m - |D_j| + |D_{j-1}|.$$

Thus the conclusion of (2.14) is true if and only if $|D_j| - |D_{j-1}| > 0$. Since $j-1 \in D$, it follows that $D_{j-1} \subsetneq D_j$, and the claim is true.

(ii) Let $j - 1 \notin D$. In this case, we have $u_{j-1} = (j-2)m$, and we conclude

$$u_j - u_{j-1} = (j-1)m - 1 - |D_j| - (j-2)m = m - 1 - |D_j| < m,$$

as desired.

Hence the interval $[\mathfrak{o}, \mathfrak{p}]$ is spherical if and only if (2.15) is satisfied. □

Corollary 2.3.11

Let $m, n > 0$, and let $\mathfrak{p} \in \mathcal{D}_n^{(m)}$ with $\mathbf{u}_{\mathfrak{p}} = (u_1, u_2, \ldots, u_n)$. Then,

$$\mu_{\mathcal{T}_n^{(m)}}(\mathfrak{p}, \mathfrak{1}) = \begin{cases} (-1)^{|D_{\mathfrak{p}}|}, & \textit{if } u_j = \mathrm{ps}_{\mathfrak{p}}(j) + 1 \textit{ for all } j \in D_{\mathfrak{p}}, \\ 0, & \textit{otherwise}, \end{cases}$$

where $D_{\mathfrak{p}}$ is the set defined just before (2.13).

Proof. By definition, we have $\mathbf{u}_{\mathfrak{1}} = (0, 0, \ldots, 0)$. Hence for $\mathfrak{p} \in \mathcal{D}_n^{(m)}$ Condition (2.14) is trivially true. Moreover, Condition (2.15) reduces to $u_j = \mathrm{ps}_{\mathfrak{p}}(j) + 1$. □

We conclude this section with the following result.

Proposition 2.3.12

Let $m, n > 0$, and define $\mathcal{S}_n^{(m)}(\mathfrak{o}) = \bigl\{\mathfrak{p} \in \mathcal{D}_n^{(m)} \mid \mu_{\mathcal{T}_n^{(m)}}(\mathfrak{o}, \mathfrak{p}) \neq 0\bigr\}$ and $\mathcal{S}_n^{(m)}(\mathfrak{1}) = \bigl\{\mathfrak{p} \in \mathcal{D}_n^{(m)} \mid \mu_{\mathcal{T}_n^{(m)}}(\mathfrak{p}, \mathfrak{1}) \neq 0\bigr\}$. Then we have

$$\bigl|\mathcal{S}_n^{(m)}(\mathfrak{o})\bigr| = 2^{n-1} = \bigl|\mathcal{S}_n^{(m)}(\mathfrak{1})\bigr|.$$

Proof. We first compute the cardinality of $\mathcal{S}_n^{(m)}(\mathfrak{o})$. Let $D \subseteq \{2, 3, \ldots, n\}$, and for $j \in D$ define $D_j = \{i < j \mid i \in D\}$. Define a sequence $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ by

$$u_j = \begin{cases} (j-1)m - 1 - |D_j|, & \text{if } j \in D, \\ (j-1)m, & \text{otherwise.} \end{cases}$$

Then, $\mathbf{u}$ is the step sequence of some $\mathfrak{p} \in \mathcal{D}_n^{(m)}$, and Corollary 2.3.10 implies that $\mu_{\mathcal{T}_n^{(m)}}(\mathfrak{o}, \mathfrak{p}) = (-1)^{|D|}$. Clearly, for each $D \subseteq \{2, 3, \ldots, n\}$, there is a unique element with this property. Hence $\bigl|\mathcal{S}_n^{(m)}(\mathfrak{o})\bigr| = 2^{n-1}$.

Now we compute the cardinality of $\mathcal{S}_n^{(m)}(\mathfrak{1})$. Let $i \in \{2,3,\ldots,n\}$, and let $\{j_1,j_2,\ldots,j_t\} \subseteq \{i+1,i+2,\ldots,n\}$ with $j_1 < j_2 < \cdots < j_t$. Set $j_0 = i$, and consider the sequence

$$\mathbf{u} = (\underbrace{0,0,\ldots,0}_{j_0-1},\underbrace{1,1,\ldots,1}_{j_1-j_0},\underbrace{2,2,\ldots,2}_{j_2-j_1},\ldots,\underbrace{t+1,t+1,\ldots,t+1}_{n-j_t+1}).$$

Since $i > 1$, this sequence is indeed the step sequence of some $\mathfrak{p} \in \mathcal{D}_n^{(m)}$, and we have $D_{\mathfrak{p}} = \{j_0,j_1,\ldots,j_t\}$. Moreover, we have $\mathrm{ps}_{\mathfrak{p}}(j_i) = i$. By construction if $u_{j-1} < u_j$, then $u_j = i+1 = \mathrm{ps}_{\mathfrak{p}}(j_i)+1$, and Corollary 2.3.11 implies that only in this case $\mu_{\mathcal{T}_n^{(m)}}(\mathfrak{p},\mathfrak{1}) = (-1)^{|D_{\mathfrak{p}}|}$. Hence we obtain

$$\left|\mathcal{S}_n^{(m)}(\mathfrak{1})\right| = 1 + \sum_{i=2}^{n} 2^{n-i} = 1 + \sum_{i=0}^{n-2} 2^i = 1 + 2^{n-1} - 1 = 2^{n-1}$$

as desired. □

2.4. A new Realization of $\mathcal{T}_n^{(m)}$ via Tuples of Dyck Paths

In this section, we define a certain subposet of the m-fold direct product of a bounded poset, the so-called m-cover poset. We use this construction to obtain a new realization of $\mathcal{T}_n^{(m)}$ in terms of m-tuples of classical Dyck paths of length $2n$. Subsequently, we consider the m-cover poset of a poset associated with the dihedral groups, and we obtain a family of "m-Tamari like" lattices for the dihedral groups. We conclude this section by discussing some drawbacks when trying to apply this construction to arbitrary Coxeter groups.

2.4.1. The m-Cover Poset. We start with a very general poset construction. Let $\mathcal{P} = (P,\leq)$ be a finite bounded poset, let $m > 0$, and consider tuples of the form

$$(\hat{0}^{l_0},p^{l_1},q^{l_2}) = (\underbrace{\hat{0},\hat{0},\ldots,\hat{0}}_{l_0},\underbrace{p,p,\ldots,p}_{l_1},\underbrace{q,q,\ldots,q}_{l_2}), \tag{2.16}$$

for some $l_0,l_1,l_2 \in \mathbb{N}$ with $l_0+l_1+l_2 = m$. The object of our interest is the following poset.

Definition 2.4.1

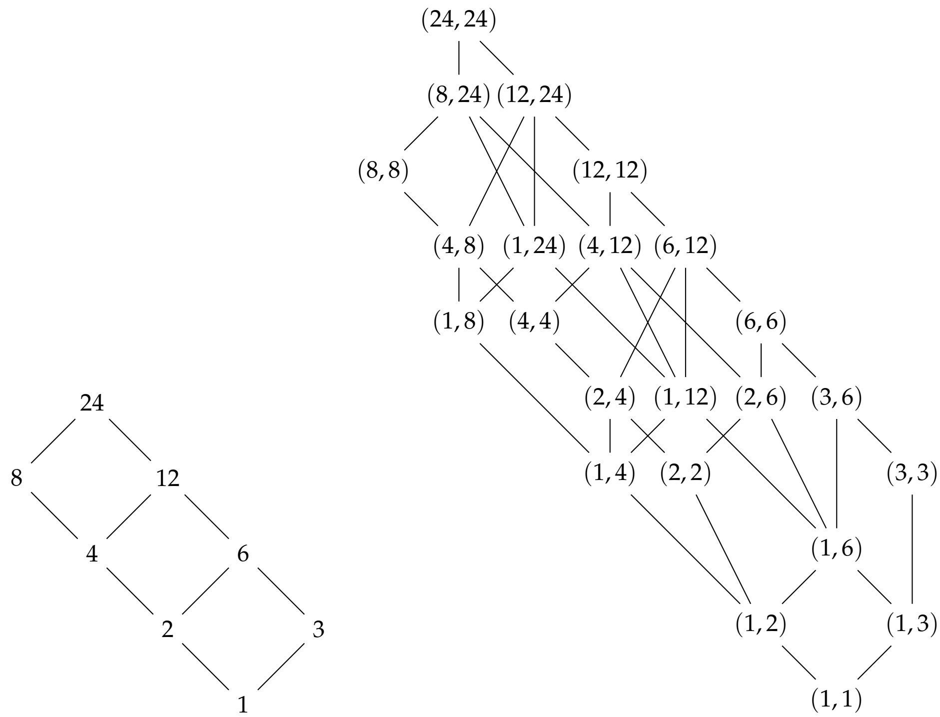
Let $\mathcal{P} = (P,\leq)$ be a bounded poset, and let $m > 0$. Consider the set

$$P^{\langle m\rangle} = \left\{(\hat{0}^{l_0},p_1^{l_1},p_2^{l_2}) \in P^m \mid p_1 \lessdot p_2, l_0+l_1+l_2 = m\right\}. \tag{2.17}$$

The poset $\mathcal{P}^{\langle m\rangle} = \left(P^{\langle m\rangle},\leq\right)$ is called the *m-cover poset of $\mathcal{P}$*, where its partial order is the componentwise partial order of $\mathcal{P}$.

Remark 2.4.2

In fact, the previous definition does not necessarily require $\mathcal{P}$ to be bounded, it would be sufficient if $\mathcal{P}$ had a least element. However, it is easy to see that $\mathcal{P}^{\langle m\rangle}$ is bounded if and only if $\mathcal{P}$ is bounded, and since we are mainly interested in the case where $\mathcal{P}^{\langle m\rangle}$ is a lattice (and hence bounded), we will use this definition.



(a) The poset $\mathcal{D}_{24}$ from Example 1.1.1.

(b) The 2-cover poset $\mathcal{D}_{24}^{\langle 2\rangle}$.

FIGURE 17. A poset and its corresponding 2-cover poset.

EXAMPLE 2.4.3

Let us again consider the poset $\mathcal{D}_{24} = (D(24), |)$ from Example 1.1.1. For the convenience of the reader, we have drawn the Hasse diagram of $\mathcal{D}_{24}$ again in Figure 17(a). If we consider $m = 2$, then the set $\big(D(24)\big)^{\langle 2\rangle}$ consists of the following 23 elements:

$$\begin{array}{llllllll} (1,1), & (1,2), & (1,3), & (1,4), & (1,6), & (1,8), & (1,12), & (1,24),\\ (2,2), & (2,4), & (2,6), & (3,3), & (3,6), & (4,4), & (4,8), & (4,12),\\ (6,6), & (6,12), & (8,8), & (8,24), & (12,12), & (12,24), & (24,24), & \end{array}$$

and the poset $\mathcal{D}_{24}^{\langle 2\rangle}$ is displayed in Figure 17(b).

Before we investigate the m-cover poset of the Tamari lattice, we start with some general properties of the m-cover poset of an arbitrary bounded poset. First of all we determine its cardinality and its length.

PROPOSITION 2.4.4

Let $\mathcal{P} = (P, \leq)$ be a bounded poset with n elements, k atoms and c cover relations. For $m > 0$, we have $\ell(\mathcal{P}^{\langle m\rangle}) = m \cdot \ell(\mathcal{P})$, and

$$\left|P^{\langle m\rangle}\right| = (c-k)\cdot\binom{m}{2} + m(n-1) + 1. \tag{2.18}$$

Proof. Suppose that $\mathcal{P}$ is a bounded poset with $\ell(\mathcal{P}) = s$, and let $\hat{0} = p_0 \lessdot p_1 \lessdot \cdots \lessdot p_s = \hat{1}$ be a maximal chain of $\mathcal{P}$. Define $\mathbf{p}_{0,m} = \left(\hat{0}^m\right)$, as well as $\mathbf{p}_{i,j} = \left(p_{i-1}^{m-j}, p_i^j\right)$ for $i \in \{1,2,\ldots,s\}$ and $j \in \{1,2,\ldots,m\}$. It is immediately clear that $\mathbf{p}_{i,j} \lessdot \mathbf{p}_{i,j+1}$ for $i \in \{1,2,\ldots,s\}$ and $j \in \{1,2,\ldots,m-1\}$, as well as $\mathbf{p}_{i-1,m} \lessdot \mathbf{p}_{i,1}$ for all $i \in \{1,2,\ldots,s\}$. Thus the chain

$$\mathbf{p}_{0,m} \lessdot \mathbf{p}_{1,1} \lessdot \mathbf{p}_{1,2} \lessdot \mathbf{p}_{1,m} \lessdot \mathbf{p}_{2,1} \lessdot \mathbf{p}_{2,2} \lessdot \cdots \lessdot \mathbf{p}_{s,m} \tag{2.19}$$

is a maximal chain in $\mathcal{P}^{\langle m\rangle}$ with length ms, which implies $\ell\left(\mathcal{P}^{\langle m\rangle}\right) \geq ms$. Since $\mathcal{P}^{\langle m\rangle}$ is a subposet of the m-fold direct product of $\mathcal{P}$ with itself, it follows that $\ell\left(\mathcal{P}^{\langle m\rangle}\right) \leq \ell\left(\mathcal{P}^m\right) = ms$, which implies the claim.

Now we want to compute the cardinality of $P^{\langle m\rangle}$. If $\mathbf{p} \in P^{\langle m\rangle}$, then it necessarily has to be of one of the following four forms:

(i) $\mathbf{p} = \left(\hat{0}^{l_0}, p_1^{l_1}, p_2^{l_2}\right)$ with $l_0, l_1, l_2 \neq 0$ and $\hat{0} \neq p_1 \lessdot p_2$. Clearly, there are $c-k$ possible choices for p_1 and p_2, and each such choice yields $\binom{m-1}{2}$ distinct elements of $P^{\langle m\rangle}$.

(ii) $\mathbf{p} = \left(p^l, q^{m-l}\right)$ with $l \in \{1,2,\ldots,m-1\}$ and $\hat{0} \neq p \lessdot q$. Again, there are $c-k$ possible choices of p and q, and each such choice yields $m-1$ distinct elements of $P^{\langle m\rangle}$.

(iii) $\mathbf{p} = \left(\hat{0}^l, p^{m-l}\right)$ with $l \in \{1,2,\ldots,m-1\}$ and $\hat{0} \neq p$. We see immediately that there are $(m-1)(n-1)$ distinct elements of this form in $P^{\langle m\rangle}$.

(iv) $\mathbf{p} = \left(p^m\right)$ with $p \in P$. There are n distinct elements of this form in $P^{\langle m\rangle}$.

If we add all these possibilities, then we obtain

$$\begin{aligned}\left|P^{\langle m\rangle}\right| &= (c-k)\binom{m-1}{2} + (c-k)(m-1) + (m-1)(n-1) + n\\ &= (c-k)\binom{m}{2} + m(n-1) + 1,\end{aligned}$$

as desired. □

Example 2.4.5

The poset $\mathcal{D}_{24}$ in Figure 17(a) consists of 8 elements, two of which are atoms, and it has 10 cover relations. The cardinality of $\mathcal{D}_{24}^{\langle 2\rangle}$ is $8 + 14 + 1 = 23$ as can be seen in Figure 17(b).

The length of $\mathcal{D}_{24}$ is 4, and a maximal chain is for instance $\{1,2,4,8,24\}$. For $m = 2$, the corresponding chain constructed in the proof of Proposition 2.4.4 is

$$\big\{(1,1),(1,2),(2,2),(2,4),(4,4),(4,8),(8,8),(8,24),(24,24)\big\},$$

and it can be checked in Figure 17(b) that this is indeed a maximal chain in $\mathcal{D}_{24}^{\langle 2\rangle}$.

Next, we will generalize the concept of join- and meet irreducible elements in a lattice to arbitrary posets. By definition, join- or meet-irreducible elements are those lattice elements that have precisely one lower or one upper cover, respectively. From this point of view, these elements can be defined without any lattice-specific terminology, so we can look for such elements in arbitrary posets as well. Hence if $\mathcal{P} = (P, \leq)$ is a poset, then we call $p \in P$ *join-irreducible* if it has precisely one lower cover, and we denote this element by $p_\star$. Moreover, we denote the set of join-irreducible elements of $\mathcal{P}$ by $\mathcal{J}(\mathcal{P})$. Dually, we call $p \in P$ *meet-irreducible* if it has precisely one upper cover, and we denote this element by $p^\star$. We denote the set of meet-irreducible elements of $\mathcal{P}$ by $\mathcal{M}(\mathcal{P})$. Let us now characterize the irreducible elements of the m-cover poset.

PROPOSITION 2.4.6

Let $\mathcal{P}$ be a bounded poset, and let $m > 0$. Then,

$$\mathcal{J}\big(\mathcal{P}^{\langle m\rangle}\big) = \Big\{\big(\hat{0}^l, p^{m-l}\big) \in P^{\langle m\rangle} \mid p \in \mathcal{J}(\mathcal{P}) \text{ and } 0 \leq l < m\Big\}, \quad \text{and}$$
$$\mathcal{M}\big(\mathcal{P}^{\langle m\rangle}\big) = \Big\{\big(p^l, (p^\star)^{m-l}\big) \in P^{\langle m\rangle} \mid p \in \mathcal{M}(\mathcal{P}) \setminus \{\hat{0}\} \text{ and } 0 < l \leq m\Big\}$$
$$\cup \Big\{\big(\hat{0}^l, \hat{1}^{m-l}\big) \mid \hat{1} \in \mathcal{J}(\mathcal{P}) \text{ and } 0 < l \leq m\Big\} \cup \Big\{\big(\hat{0}^m\big) \mid \hat{0} \in \mathcal{M}(\mathcal{P})\Big\}.$$

PROOF. Let $\mathbf{p} = \big(\hat{0}^{l_0}, p_1^{l_1}, p_2^{l_2}\big) \in P^{\langle m\rangle}$ with $\hat{0} \neq p \lessdot q$. First suppose that $\mathbf{p} \in \mathcal{J}\big(\mathcal{P}^{\langle m\rangle}\big)$. If $l_1 > 0$ and $l_2 > 0$, then it follows that the elements $\mathbf{p}' = \big(\hat{0}^{l_0+1}, p_1^{l_1-1}, p_2^{l_2}\big)$ and $\mathbf{p}'' = \big(\hat{0}^{l_0}, p_1^{l_1+1}, p_2^{l_2-1}\big)$ are both lower covers of $\mathbf{p}$ in $\mathcal{P}^{\langle m\rangle}$, contradicting the assumption that $\mathbf{p}$ is join-irreducible. Without loss of generality, we can assume that $\mathbf{p} = \big(\hat{0}^{l_0}, p_1^{l_1}\big)$. If $l_1 = 0$, then $\mathbf{p}$ is the least element of $\mathcal{P}^{\langle m\rangle}$, and thus not join-irreducible, contradicting the choice of $\mathbf{p}$. Now for every element $\bar{p} \in P$ with $\bar{p} \lessdot p_1$, the element $\bar{\mathbf{p}} = \big(\hat{0}^{l_0}, \bar{p}, p_1^{l_1-1}\big)$ is a lower cover of $\mathbf{p}$, which implies the claim.

Now suppose that $\mathbf{p} \in \mathcal{M}\big(\mathcal{P}^{\langle m\rangle}\big)$. If $l_0 > 0$ and $l_1 > 0$, then it follows that the elements $\mathbf{p}' = \big(\hat{0}^{l_0-1}, p_1^{l_1+1}, p_2^{l_2}\big)$ and $\mathbf{p}'' = \big(\hat{0}^{l_0}, p_1^{l_1-1}, p_2^{l_2+1}\big)$ are both upper covers of $\mathbf{p}$ in $\mathcal{P}^{\langle m\rangle}$, contradicting the assumption that $\mathbf{p}$ is meet-irreducible. (In the special case where $l_1 = 0$ and $l_2 > 0$, we can apply the same reasoning.) Hence we have either $l_0 = 0$ or one of l_1 and l_2 is zero.

First let $l_0 = 0$, and thus $\mathbf{p} = \big(p_1^{l_1}, p_2^{l_2}\big)$. Clearly the element $\mathbf{p}'' = \big(p_1^{l_1-1}, p_2^{l_2+1}\big)$ satisfies $\mathbf{p} \lessdot \mathbf{p}''$. Assume that $p_1 \notin \mathcal{M}(\mathcal{P})$. Then it follows immediately that $p_2 \neq \hat{1}$, because otherwise p_1 is a coatom, and thus clearly meet-irreducible, contradicting the assumption. Thus we can choose an upper cover q_2 of p_2 in $\mathcal{P}$, and some upper cover q_1 of p_1 in $\mathcal{P}$ with $q_1 \neq p_2$. It follows that $q_1 \neq \hat{1}$, and we distinguish three cases:

(i) If $q_1 \leq q_2$, then there exists a chain $q_1 = w_1 \lessdot w_2 \cdots \lessdot w_k \lessdot q_2$ in $\mathcal{P}$, and we have $p_2 \not\leq w_k$, because otherwise we would obtain a contradiction to $p_2 \lessdot q_2$. Consider the element $\bar{\mathbf{p}} = \big(w_k^{l_1}, q_2^{l_2}\big)$, which satisfies $\mathbf{p} \leq \bar{\mathbf{p}}$. Suppose that there is some element $\mathbf{q} \in P^{\langle m\rangle}$ with $\mathbf{p} \lessdot \mathbf{q} \leq \bar{\mathbf{p}}$. Since $p_2 \not\leq w_k$ it follows that $\mathbf{q} = \big(x^{l_1}, y^{l_2}\big)$ for $p_1 \leq x \leq w_k$ and $p_2 \leq y \leq q_2$. If $y = p_2$, then necessarily $x = p_1$, and we obtain $\mathbf{q} = \mathbf{p}$, which contradicts the choice of $\mathbf{q}$. Hence $y = q_2$, and since x is a lower cover of y it follows that $x = w_k$, which implies $\mathbf{q} = \bar{\mathbf{p}}$. Hence $\mathbf{p} \lessdot \mathbf{q}$. However, since $\bar{\mathbf{p}} \neq \mathbf{p}''$ we obtain a contradiction to $\mathbf{p}$ being meet-irreducible in $\mathcal{P}^{\langle m\rangle}$.

(ii) If $q_2 \leq q_1$, then the reasoning is analogous to (i).

(iii) If $q_1 \parallel q_2$, then, since $\mathcal{P}$ is bounded, there exists a (not necessarily unique) minimal element $w \in P$ with $q_1, q_2 \leq w$, and there exist chains $q_1 = u_1 \lessdot u_2 \lessdot \cdots \lessdot u_k \lessdot w$ and $q_2 = v_1 \lessdot v_2 \lessdot \cdots \lessdot v_l \lessdot w$. Consider the element $\bar{\mathbf{p}} = \big(u_k^{l_1}, w^{l_2}\big)$, which satisfies $\mathbf{p} \leq \bar{\mathbf{p}}$. Again, suppose that there is some element $\mathbf{q} \in P^{\langle m\rangle}$ with $\mathbf{p} \lessdot \mathbf{q} \leq \bar{\mathbf{p}}$. The minimality of w ensures that $p_2 \not\leq u_k$, and it follows that $\mathbf{q} = \big(x^{l_1}, y^{l_2}\big)$ for $p_1 \leq x \leq u_k$ and $p_2 \leq y \leq_P w$. The minimality of w also ensures that $u_i \parallel v_j$ for all $i \in \{1, 2, \ldots, k\}$ and $j \in \{1, 2, \ldots, l\}$. Since x is a lower cover of y and $\mathbf{q} \neq \mathbf{p}$, it follows that $x = u_k$ and $y = w$, which implies $\mathbf{q} = \bar{\mathbf{p}}$. Thus $\mathbf{p} \lessdot \bar{\mathbf{p}}$. However, since $\bar{\mathbf{p}} \neq \mathbf{p}''$ we obtain a contradiction to $\mathbf{p}$ being meet-irreducible in $\mathcal{P}^{\langle m\rangle}$.

Hence if $l_0 = 0$, then it follows that $p_1 \in \mathcal{M}(\mathcal{P})$ and $p_2 = p_1^\star$.

Now suppose that $l_0 > 0$, and thus that $l_1 = 0$ or $l_2 = 0$. Without loss of generality, we can write $\mathbf{p} = \big(\hat{0}^{l_0}, p_1^{l_1}\big)$. If $l_1 = 0$, then every atom of $\mathcal{P}$ yields an upper cover of $\mathbf{p}$, and hence $\mathbf{p} \in \mathcal{M}(\mathcal{P}^{\langle m\rangle})$ if and only if $\hat{0} \in \mathcal{M}(\mathcal{P})$. Now let $l_1 > 0$. If $p_1 \neq \hat{1}$, then the element $\mathbf{p}' = (\hat{0}^{l_1-1}, p_1^{l_2+1})$ satisfies $\mathbf{p} \lessdot \mathbf{p}'$. Moreover, for every upper cover q of p_1 in $\mathcal{P}$, the element $(\hat{0}^{l_0}, p_1^{l_2-1}, q)$ is an upper cover of $\mathbf{p}$ different from $\mathbf{p}'$, which is a contradiction to $\mathbf{p}$ being meet-irreducible in $\mathcal{P}^{\langle m\rangle}$. If $p_1 = \hat{1}$, then for every coatom c of $\mathcal{P}$ the element $(\hat{0}^{l_0-1}, c, \hat{1}^{l_1})$ is an upper cover of $\mathbf{p}$. Hence if $l_0 > 0$, then it follows that either $\mathbf{p} = (\hat{0}^m)$ provided that $\hat{0} \in \mathcal{M}(\mathcal{P})$ or $\mathbf{p} = (\hat{0}^{l_0}, \hat{1}^{l_1})$ provided that $\hat{1} \in \mathcal{J}(\mathcal{P})$. □

In view of Propositions 2.4.4 and 2.4.6, we can determine the posets for which every m-cover poset is extremal, i.e. where $\left|\mathcal{J}(\mathcal{P}^{\langle m\rangle})\right| = \ell(\mathcal{P}^{\langle m\rangle}) = \left|\mathcal{M}(\mathcal{P}^{\langle m\rangle})\right|$.

Corollary 2.4.7

Let $\mathcal{P}$ be a bounded extremal poset, with $\ell(\mathcal{P}) = k$. Then, $\mathcal{P}^{\langle m\rangle}$ is extremal for every $m > 0$ if and only if either $\hat{0} \in \mathcal{J}(\mathcal{P})$ and $\hat{1} \in \mathcal{M}(\mathcal{P})$ or $\hat{0} \notin \mathcal{J}(\mathcal{P})$ and $\hat{1} \notin \mathcal{M}(\mathcal{P})$.

Proof. Proposition 2.4.4 implies that $\ell(\mathcal{P}^{\langle m\rangle}) = mk$, and it follows from the first part of Proposition 2.4.6 that $\left|\mathcal{J}(\mathcal{P}^{\langle m\rangle})\right| = m\left|\mathcal{J}(\mathcal{P})\right| = mk$. Thus it remains to determine the cardinality of the set of meet-irreducibles of $\mathcal{P}^{\langle m\rangle}$. If $\hat{0} \in \mathcal{M}(\mathcal{P})$ and $\hat{1} \notin \mathcal{J}(\mathcal{P})$, then the second part of Proposition 2.4.6 implies $\left|\mathcal{M}(\mathcal{P}^{\langle m\rangle})\right| = m(k-1)+1 < mk$ unless $m = 1$. Analogously, if $\hat{0} \notin \mathcal{M}(\mathcal{P})$ and $\hat{1} \in \mathcal{J}(\mathcal{P})$, then the second part of Proposition 2.4.6 implies $\left|\mathcal{M}(\mathcal{P}^{\langle m\rangle})\right| = (m+1)k > mk$. On the other hand if $\hat{0} \in \mathcal{M}(\mathcal{P})$ and $\hat{1} \in \mathcal{J}(\mathcal{P})$ or $\hat{0} \notin \mathcal{M}(\mathcal{P})$ and $\hat{1} \notin \mathcal{J}(\mathcal{P})$, then the second part of Proposition 2.4.6 implies $\left|\mathcal{M}(\mathcal{P}^{\langle m\rangle})\right| = mk$ as desired. □

If we take a closer look at Figure 17, then we notice that the poset $\mathcal{D}_{24}$ is a lattice while its 2-cover poset $\mathcal{D}_{24}^{\langle 2\rangle}$ is not, since for instance the elements $(4,12)$ and $(6,12)$ do not have a meet. Since by definition the poset $\mathcal{P}^{\langle m\rangle}$ is an interval of $\mathcal{P}^{\langle m+1\rangle}$ for every bounded poset $\mathcal{P}$, it follows that no m-cover poset of $\mathcal{D}_{24}$ is a lattice, unless $m = 1$. The next proposition characterizes the bounded posets, whose m-cover posets are always lattices.

Proposition 2.4.8

Let $\mathcal{P} = (P, \leq)$ be a bounded poset. The m-cover poset $\mathcal{P}^{\langle m\rangle}$ is a lattice for all $m > 0$ if and only if $\mathcal{P}$ is a lattice and for all $p, q \in P$ we have $p \wedge q \in \{\hat{0}, p, q\}$.

Proof. Suppose that $\mathcal{P}$ is a lattice, and suppose that for every $p, q \in P$, we have $p \wedge q \in \{\hat{0}, p, q\}$. We want to show first that $\mathcal{P}^{\langle m\rangle}$ is a lattice again. Let $\mathbf{p} = (\hat{0}^{k_0}, p_1^{k_1}, p_2^{k_2})$ and $\mathbf{q} = (\hat{0}^{l_0}, q_1^{l_1}, q_2^{l_2})$. We show that the componentwise meet of $\mathbf{p}$ and $\mathbf{q}$, denoted by $\mathbf{z}$, is again contained in $P^{\langle m\rangle}$, and since $\mathcal{P}^{\langle m\rangle}$ is a subposet of $\mathcal{P}^m$ it follows that $\mathbf{z}$ has to be the meet of $\mathbf{p}$ and $\mathbf{q}$ in $\mathcal{P}^{\langle m\rangle}$. We essentially have two choices for $\mathbf{z}$, depending on the values of k_0, k_1, k_2 and l_0, l_1, l_2:

$$\mathbf{z} = \left(\hat{0}^{s_0}, (p_1 \wedge q_1)^{s_1}, (p_1 \wedge q_2)^{s_2}, (p_2 \wedge q_2)^{s_3}\right), \quad \text{or} \tag{2.20}$$

$$\mathbf{z} = \left(\hat{0}^{s_0}, (p_1 \wedge q_1)^{s_1}, (p_2 \wedge q_1)^{s_2}, (p_2 \wedge q_2)^{s_3}\right), \tag{2.21}$$

for suitable $s_0, s_1, s_2, s_3 \in \{0, 1, \ldots, m\}$, and we distinguish three cases.

(i) Let $p_1 \wedge q_1 = \hat{0}$. Here we need to distinguish three more cases:

(ia) Let $p_1 \wedge q_2 = \hat{0}$. If $\mathbf{z}$ is of the form (2.20), then it follows immediately that $\mathbf{z} \in P^{\langle m\rangle}$. So,

suppose that $\mathbf{z}$ is of the form (2.21). If $q_1 \leq p_2$, then $q_1 \leq p_2 \wedge q_2 \leq q_2$, which implies with $q_1 \lessdot q_2$ that $\mathbf{z} \in P^{\langle m\rangle}$. If $p_2 \leq q_1$, then it follows immediately that $\mathbf{z} \in P^{\langle m\rangle}$. If $q_1 \parallel p_2$, then $q_1 \wedge p_2 = \hat{0}$ by assumption and again $\mathbf{z} \in P^{\langle m\rangle}$.
(ib) Let $p_1 \wedge q_2 = p_1$. Then, both p_2 and q_2 are upper bounds for p_1, and hence $p_1 \leq p_2 \wedge q_2$. If $p_2 \parallel q_2$, then $p_1 = \hat{0} = q_1$, and it follows that $\mathbf{z} = \left(\hat{0}^m\right) \in P^{\langle m\rangle}$. If $q_2 \leq p_2$, then $p_1 = q_2$, and it follows that $\mathbf{z} = \left(\hat{0}^{s_0+s_1}, p_1^{s_2+s_3}\right) \in P^{\langle m\rangle}$ or $\mathbf{z} = \left(\hat{0}^{s_0+s_1}, q_1^{s_2}, q_2^{s_3}\right) \in P^{\langle m\rangle}$. If $p_2 \leq q_2$, then we have either $p_2 \parallel q_1$ or $p_2 = q_1$. In both cases, if $\mathbf{z}$ is of the form (2.20), then we have $\mathbf{z} = \left(\hat{0}^{s_0+s_1}, p_1^{s_2}, p_2^{s_3}\right) \in P^{\langle m\rangle}$. If $\mathbf{z}$ is of the form (2.21), then the first case yields $\mathbf{z} = \left(\hat{0}^{s_0+s_1+s_2}, p_2^{s_3}\right) \in P^{\langle m\rangle}$, and the second case yields $\mathbf{z} = \left(\hat{0}^{s_0+s_1}, p_2^{s_2+s_3}\right) \in P^{\langle m\rangle}$.
(ic) Let $p_1 \wedge q_2 = q_2$. This works analogously to (ib).

(ii) Let $p_1 \wedge q_1 = p_1$. Then, it follows by assumption that either $p_2 \leq q_1$ or $p_1 = \hat{0}$. In the first case, we have $\mathbf{z} = \left(\hat{0}^{s_0}, p_1^{s_1+s_2}, p_2^{s_3}\right) \in P^{\langle m\rangle}$ if $\mathbf{z}$ is of the form (2.20), or $\mathbf{z} = \left(\hat{0}^{s_0}, p_1^{s_1}, p_2^{s_2+s_3}\right) \in P^{\langle m\rangle}$ if $\mathbf{z}$ is of the form (2.21). In the second case, we have $\mathbf{z} = \left(\hat{0}^{s_0+s_1+s_2}, (p_2 \wedge q_2)^{s_3}\right) \in P^{\langle m\rangle}$ if $\mathbf{z}$ is of the form (2.20). Thus it remains to consider the case where $p_1 = \hat{0}$, and $\mathbf{z}$ is of the form (2.21). Then, we have either $p_2 \wedge q_1 = \hat{0}$ (which implies $\mathbf{z} = \left(\hat{0}^m\right) \in P^{\langle m\rangle}$), or $p_2 \wedge q_1 = p_2$ (which implies $\mathbf{z} = \left(\hat{0}^{s_0+s_1}, p_2^{s_2+s_3}\right) \in P^{\langle m\rangle}$), or $p_2 \wedge q_1 = q_1$ (which implies $\mathbf{z} = \left(\hat{0}^{s_0+s_1}, q_1^{s_2}, q_2^{s_3}\right) \in P^{\langle m\rangle}$).

(iii) Let $p_1 \wedge q_1 = q_1$. This works analogously to (ii).

Hence every two elements $\mathbf{p}, \mathbf{q} \in P^{\langle m\rangle}$ have a meet in $\mathcal{P}^{\langle m\rangle}$, and since $\mathcal{P}^{\langle m\rangle}$ is finite and bounded, it is a classical lattice-theoretic result that $\mathcal{P}^{\langle m\rangle}$ is a lattice.

We prove the converse argument by contradiction. Since $\mathcal{P}$ is an interval in $\mathcal{P}^{\langle m\rangle}$, it follows immediately that if $\mathcal{P}$ is no lattice, then $\mathcal{P}^{\langle m\rangle}$ cannot be a lattice as well. So suppose that $\mathcal{P}$ is a lattice, and suppose further that there exist two elements $p, q \in P$, with $p \wedge q = z \notin \{\hat{0}, p, q\}$. We explicitly construct two elements $\mathbf{p}, \mathbf{q} \in P^{\langle m\rangle}$ that do not have a meet in $\mathcal{P}^{\langle m\rangle}$. Without loss of generality, we may assume that if $\bar{p}, \bar{q} \in P$ with $p \leq \bar{p}$ and $q \leq \bar{q}$ satisfy $\bar{p} \wedge \bar{q} \notin \{\hat{0}, \bar{p}, \bar{q}\}$, then $p = \bar{p}$ or $q = \bar{q}$. It follows further immediately that neither $p = \hat{1}$ nor $q = \hat{1}$. Hence we can find elements $p', q' \in P$ with $p \lessdot p'$ and $q \lessdot q'$, and it follows that $p' \wedge q' \in \{\hat{0}, p', q'\}$. Moreover, since $\hat{0} \neq z \leq p' \wedge q'$, it follows that $p' \leq q'$ or $q' \leq p'$, and we assume without loss of generality that $p' \leq q'$.

On the one hand, consider the elements $\mathbf{p} = \left(p, (q')^{m-1}\right)$ and $\mathbf{q} = \left(q, (q')^{m-1}\right)$, and on the other hand, the consider elements $\mathbf{w}_1 = \left(\hat{0}, (p')^{m-1}\right)$ and $\mathbf{w}_2 = \left(z, (z')^{m-1}\right)$, where z' satisfies $z \lessdot z' \leq p$. Then, we have $\mathbf{w}_1, \mathbf{w}_2 \leq \mathbf{p}, \mathbf{q}$, and both $\mathbf{p}$ and $\mathbf{q}$ as well as $\mathbf{w}_1$ and $\mathbf{w}_2$ are mutually incomparable. The only candidate for an element that would be larger than $\mathbf{w}_1$ and $\mathbf{w}_2$ and at the same time smaller than $\mathbf{p}$ and $\mathbf{q}$ is $\left(z, (p')^{m-1}\right)$, which does, however, not belong to $P^{\langle m\rangle}$, since $z < p < p'$. Hence $\mathbf{p}$ and $\mathbf{q}$ do not have a meet in $\mathcal{P}^{\langle m\rangle}$, which implies that $\mathcal{P}^{\langle m\rangle}$ is not a lattice. $\square$

REMARK 2.4.9

The proof of Proposition 2.4.8 implies that if $\mathcal{P}^{\langle m\rangle}$ is a lattice, then the meet of two elements $\mathbf{p}, \mathbf{q} \in P^{\langle m\rangle}$ is their componentwise meet. However, we can find simple examples that the same is not true for joins. (Consider for instance the pentagon lattice and $m = 2$.) Hence in general $\mathcal{P}^{\langle m\rangle}$ is a meet-sublattice of $\mathcal{P}^m$.

The condition when every m-cover poset of a given bounded poset is a lattice can be reformulated in the following way.

Corollary 2.4.10

Let $\mathcal{P}$ be a bounded poset. The m-cover poset $\mathcal{P}^{\langle m\rangle}$ is a lattice for all $m > 0$ if and only if the Hasse diagram of $\mathcal{P}$ with $\hat{0}$ removed is a tree rooted at $\hat{1}$.

Proof. Let H denote the Hasse diagram of $\mathcal{P}$ with $\hat{0}$ removed, let $p, q \in P$ with $p \parallel q$, and let $m > 1$. If H is a tree rooted at $\hat{1}$, then by definition it does not contain a cycle. It is easy to check that $\mathcal{P}$ is a lattice and that $p \wedge q = \hat{0}$.

Conversely, suppose that $\mathcal{P}^{\langle m\rangle}$ is a lattice, and that $p \wedge q = \hat{0}$. If H is not a tree, then it must contain a cycle, and we can assume without loss of generality that p and q are incomparable and belong to this cycle. This implies, however, that there is an element $z \in H$ with $z \leq p \wedge q = \hat{0}$. This contradicts $\hat{0} \notin H$, and the proof is completed. □

Remark 2.4.11

The posets which occur in Proposition 2.4.8 and Corollary 2.4.10 are in principle the *chord posets* defined in [68]. More precisely, if $\mathcal{P}$ is a poset such that for all $m > 0$ the m-cover poset $\mathcal{P}^{\langle m\rangle}$ is a lattice, then the dual of the proper part of $\mathcal{P}$ is a chord poset. By definition, chord posets are in a natural bijection with Dyck paths. We think that this is a nice coincidence, since we encountered m-cover posets while studying the m-Tamari lattices, and it turns out that there is this interesting structural correspondence between both objects. From this point of view, the mysterious connection between the m-Tamari lattice and the m-cover poset of the Tamari lattice that we exhibit in Section 2.4.3 might perhaps not be too mysterious.

In view of Corollary 2.4.10, we can simplify the formula for the cardinality of $\mathcal{P}^{\langle m\rangle}$ from Proposition 2.4.4 in the lattice case.

Corollary 2.4.12

Let $\mathcal{P} = (P, \leq)$ be a bounded poset with n elements such that $\mathcal{P}^{\langle m\rangle}$ is a lattice for all $m > 0$. If $n > 1$, then

$$\left|P^{\langle m\rangle}\right| = n\binom{m+1}{2} - m^2 + 1.$$

Proof. Corollary 2.4.10 states that the Hasse diagram of $\mathcal{P}$ with $\hat{0}$ removed is a tree with $n-1$ nodes. Suppose that H has k leaves. Each leaf of H is an atom of $\mathcal{P}$, and since H has $n-1$ nodes, it follows that $\mathcal{P}$ has $n-2+k$ cover relations. If we plug these values in (2.18), then we obtain

$$\left|P^{\langle m\rangle}\right| = (n-2+k-k)\binom{m}{2} + m(n-1) + 1 = n\binom{m+1}{2} - m^2 + 1,$$

as desired. □

Besides their application within the context of the Tamari lattices, the m-cover posets are interesting posets in their own right, and we have obtained further structural and topological results in [67]. The precise statement of these results would lead too far from the central topic of this thesis, so we refer the interested reader to [67, Section 5].

2.4.2. The Strip-Decomposition. It is our overall goal in the remainder of this chapter to investigate the connection between the m-Tamari lattices and the m-cover posets of the classical Tamari lattices, and we will do that in Section 2.4.3. However, we need some more preparation. By definition, the m-cover poset of $\mathcal{T}_n$ consists of m-tuples of Dyck paths of length $2n$. In this section we introduce a new decomposition of m-Dyck paths of length $(m+1)n$ into m-tuples of Dyck paths of length $2n$. This decomposition plays a key role in Section 2.4.3, and we start right away with the definition.

DEFINITION 2.4.13

Let $\mathfrak{p} \in \mathcal{D}_n^{(m)}$ with associated height sequence $\mathbf{h}_\mathfrak{p} = (h_1, h_2, \ldots, h_{mn})$. For $i \in \{1, 2, \ldots, m\}$ let $\mathfrak{q}_i$ denote the Dyck path whose height sequence is $\mathbf{h}_{\mathfrak{q}_i} = (h_i, h_{i+m}, \ldots, h_{i+(n-1)m})$. The sequence $\delta(\mathfrak{p}) = (\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_m)$ is called the *strip-decomposition of* $\mathfrak{p}$.

The fact that δ is indeed well-defined can be seen easily using (2.3) and (2.4). Moreover, we can explicitly describe the inverse of δ by using Lemma 2.2.2. For the record we state this as a lemma.

LEMMA 2.4.14

The map $\delta : \mathcal{D}_n^{(m)} \to (\mathcal{D}_n)^m$ *from Definition 2.4.13 is well-defined and injective.*

EXAMPLE 2.4.15

Consider the 5-Dyck path $\mathfrak{p}$ of length 36 shown in Figure 18(a). Its height sequence is

$$\mathbf{h}_\mathfrak{p} = (1,1,2,3,3,3,3,3,4,4,4,4,4,4,5,5,5,5,5,5,5,5,5,6,6,6,6,6,6,6).$$

The strip-decomposition of $\mathfrak{p}$ yields the following five sequences

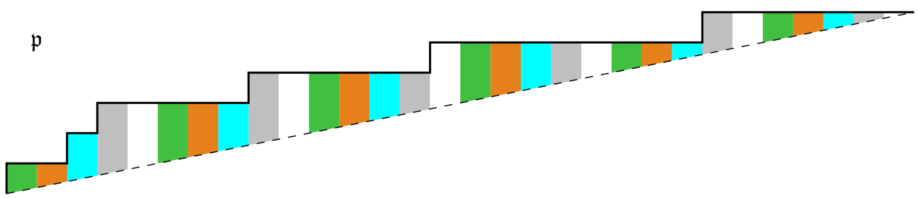
$$\begin{aligned} \mathbf{h}_{\mathfrak{q}_1} &= (1,3,4,5,5,6),\\ \mathbf{h}_{\mathfrak{q}_2} &= (1,3,4,5,5,6),\\ \mathbf{h}_{\mathfrak{q}_3} &= (2,3,4,5,5,6),\\ \mathbf{h}_{\mathfrak{q}_4} &= (3,4,4,5,6,6),\\ \mathbf{h}_{\mathfrak{q}_5} &= (3,4,5,5,6,6), \end{aligned}$$

which are the height sequences of the Dyck paths $\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_5$ of length 12 shown in Figure 18(b).

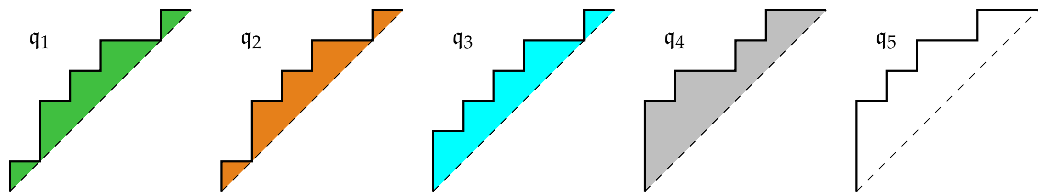
If $\mathfrak{q}, \mathfrak{q}' \in \mathcal{D}_n$ have height sequences $\mathbf{h}_\mathfrak{q} = (h_1, h_2, \ldots, h_n)$ and $\mathbf{h}_{\mathfrak{q}'} = (h'_1, h'_2, \ldots, h'_n)$, then we say that $\mathfrak{q}'$ *dominates* $\mathfrak{q}$ if $h'_i \geq h_i$ for all $i \in \{1, 2, \ldots, n\}$, and we will usually write $\mathfrak{q} \leq_{\mathrm{dom}} \mathfrak{q}'$. In other words, the two Dyck paths $\mathfrak{q}$ and $\mathfrak{q}'$ never cross, but might share some common edges. Then, we call the partial order $\leq_{\mathrm{dom}}$ the *dominance order on* $\mathcal{D}_n$. The poset $\left(\mathcal{D}_n, \leq_{\mathrm{dom}}\right)$ has been investigated for instance in [10, 48–52, 84]. According to [103] an *increasing* m*-fan of Dyck paths* is an m-tuple $(\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_m)$ of Dyck paths of length $2n$ satisfying $\mathfrak{q}_1 \leq_{\mathrm{dom}} \mathfrak{q}_2 \leq_{\mathrm{dom}} \cdots \leq_{\mathrm{dom}} \mathfrak{q}_m$.

LEMMA 2.4.16

If $\mathfrak{p} \in \mathcal{D}_n^{(m)}$, *then* $\delta(\mathfrak{p})$ *is an increasing* m*-fan of Dyck paths of length* $2n$.



(a) A 5-Dyck path $\mathfrak{p}$ of length 36.



(b) The strip-decomposition of $\mathfrak{p}$.

FIGURE 18. The strip-decomposition of a 5-Dyck path of length 36.

PROOF. Let $\mathfrak{p} \in \mathcal{D}_n^{(m)}$ with associated height sequence $\mathbf{h}_{\mathfrak{p}} = (h_1, h_2, \ldots, h_{mn})$, and let $\delta(\mathfrak{p}) = (\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_m)$. For $i \in \{1, 2, \ldots, m\}$, let $\mathbf{h}_{\mathfrak{q}_i} = \big(h_1^{(i)}, h_2^{(i)}, \ldots, h_m^{(i)}\big)$ denote the height sequence of $\mathfrak{q}_i$. Then, for $i, j \in \{1, 2, \ldots, m\}$ with $i < j$, and $k \in \{1, 2, \ldots, m\}$, it follows from (2.3) and Definition 2.4.13 that $h_k^{(i)} = h_{k+(i-1)m} \leq h_{k+(j-1)m} = h_k^{(j)}$, as desired. $\square$

EXAMPLE 2.4.17

Let $m = 2$ and $n = 3$. Figure 19 shows the twelve 2-Dyck paths of length 9 and their corresponding strip-decomposition. We notice that in total there are fourteen increasing 2-fans of Dyck paths of length 6. The remaining two increasing 2-fans of Dyck paths of length 6 are displayed in Figure 20, and it is indicated why these do not correspond to 2-Dyck paths of length 9.

The next lemma characterizes $\delta\big(\mathcal{D}_n^{(m)}\big)$.

LEMMA 2.4.18

Let $(\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_m)$ be an increasing m-fan of Dyck paths of length $2n$ with associated height sequences $\mathbf{h}_{\mathfrak{q}_j} = \big(h_1^{(j)}, h_2^{(j)}, \ldots, h_n^{(j)}\big)$ for $j \in \{1, 2, \ldots, m\}$. Then, $(\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_m)$ induces an m-Dyck path of length $(m+1)n$ via δ^{-1} if and only if $h_i^{(k)} \leq h_{i+1}^{(j)}$ for all $i \in \{1, 2, \ldots, n-2\}$ and for all $k > j$.

PROOF. Let us first consider the case $m = 2$. Then, δ^{-1} constructs a sequence $\mathbf{h} = (h_1, h_2, \ldots, h_{2n})$ with $h_{2i-1} = h_i^{(1)}$ and $h_{2i} = h_i^{(2)}$ for all $i \in \{1, 2, \ldots, n\}$. Since $(\mathfrak{q}_1, \mathfrak{q}_2)$ is increasing we obtain $h_i^{(1)} \leq h_i^{(2)}$ for all $i \in \{1, 2, \ldots, n\}$. Assume that for all $i \in \{1, 2, \ldots, n-2\}$, we have $h_i^{(2)} \leq h_{i+1}^{(1)}$. Since by definition $h_n^{(1)} = h_n^{(2)} = n$ we obtain $h_1^{(1)} \leq h_1^{(2)} \leq h_2^{(1)} \leq \cdots \leq h_n^{(2)}$, and thus $\mathbf{h}$ satisfies (2.3). Moreover, if we apply (2.4) to $\mathbf{h}_{\mathfrak{q}_1}$ and $\mathbf{h}_{\mathfrak{q}_2}$, then we obtain

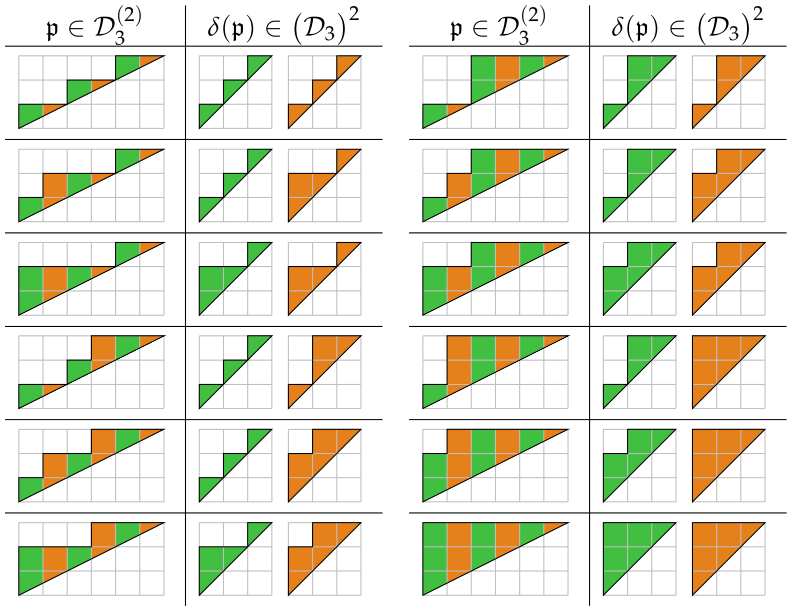


FIGURE 19. The twelve 2-Dyck paths of length 9 and the corresponding increasing 2-fans of Dyck paths of length 6.

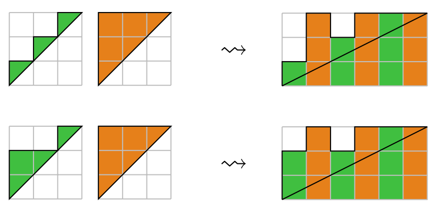


FIGURE 20. The two increasing 2-fans of Dyck paths of length 9 that do not produce a valid 2-Dyck path of length 6.

$h_{2i-1} = h_i^{(1)} \geq i = \lceil \frac{2i-1}{2} \rceil$ and $h_{2i} = h_i^{(2)} \geq i = \lceil \frac{2i}{2} \rceil$. Thus $\mathbf{h}$ satisfies (2.4) itself, and must be the height sequence of some 2-Dyck path $\mathfrak{p}$ of length $3n$.

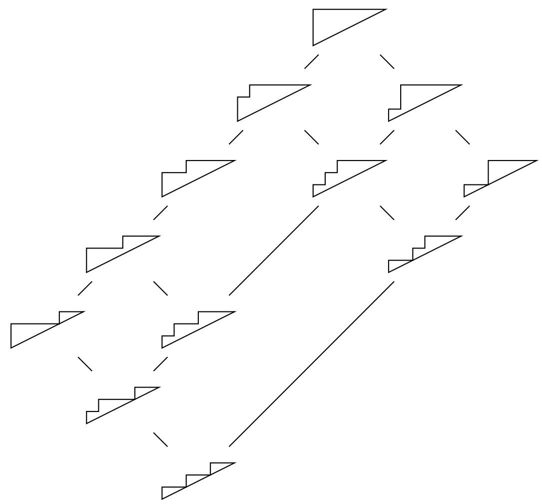
Conversely, let $\mathfrak{p} \in \mathcal{D}_n^{(2)}$ have height sequence $\mathbf{h}_{\mathfrak{p}} = (h_1, h_2, \ldots, h_{2n})$, and let $\delta(\mathfrak{p}) = (\mathfrak{q}_1, \mathfrak{q}_2)$. It follows from Lemma 2.4.16 that $(\mathfrak{q}_1, \mathfrak{q}_2)$ is indeed an increasing 2-fan of Dyck paths of length $2n$. By construction, the height sequences of $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are $\mathbf{h}_{\mathfrak{q}_1} = (h_1, h_3, \ldots, h_{2n-1})$ and $\mathbf{h}_{\mathfrak{q}_2} = (h_2, h_4, \ldots, h_{2n})$, and with (2.3) follows $h_{2i} \leq h_{2i+1}$ as desired.

The reasoning for $m > 2$ is exactly analogous. $\square$

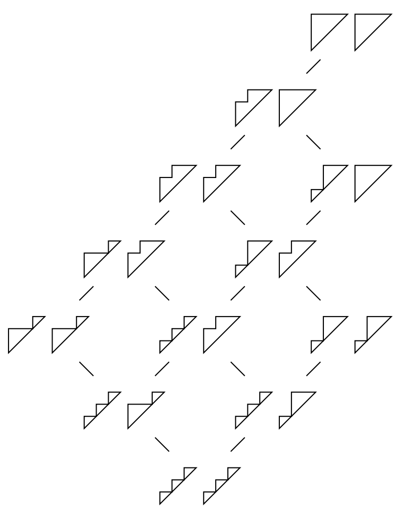
Recall that given two posets $\mathcal{P} = (P, \leq_P)$ and $\mathcal{Q} = (Q, \leq_Q)$, a map $f : P \to Q$ is called *order-preserving* if for every $p, p' \in P$ with $p \leq_P p'$ we have $f(p) \leq_Q f(p')$. We have the following property of δ.

LEMMA 2.4.19

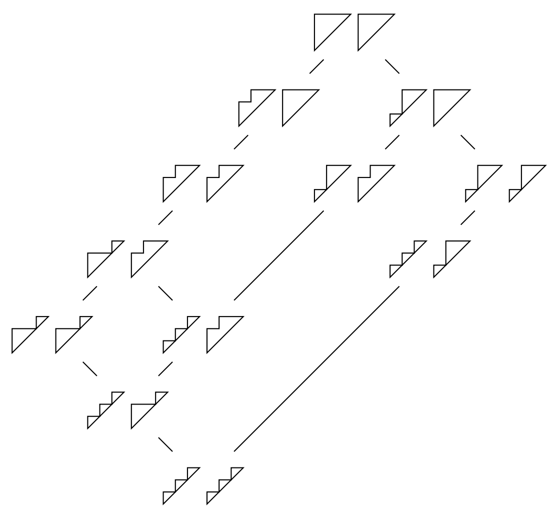
The map δ is an order-preserving map from $\left(\mathcal{D}_n^{(m)}, \leq_{\mathrm{rot}}\right)$ to $\left(\delta(\mathcal{D}_n^{(m)}), \leq_{\mathrm{dom}}\right)$.



(a) The poset $\left(\mathcal{D}_3^{(2)}, \leq_{\mathrm{rot}}\right)$.



(b) The poset $\left(\delta\left(\mathcal{D}_3^{(2)}\right), \leq_{\mathrm{dom}}\right)$.



(c) The poset $\left(\delta\left(\mathcal{D}_3^{(2)}\right), \leq_{\mathrm{rot}}\right)$.

FIGURE 21. Illustration of dominance and rotation order on $\delta\left(\mathcal{D}_3^{(2)}\right)$.

PROOF. We have to show that for $\mathfrak{p}, \mathfrak{p}' \in \mathcal{D}_n^{(m)}$ with $\delta(\mathfrak{p}) = (\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_m)$ and $\delta(\mathfrak{p}') = (\mathfrak{q}_1', \mathfrak{q}_2', \ldots, \mathfrak{q}_m')$, we have $\mathfrak{q}_i \leq_{\mathrm{dom}} \mathfrak{q}_i'$ for all $i \in \{1, 2, \ldots, m\}$. This, however, follows immediately from Lemma 2.2.6. $\square$

Figures 21(a) and 21(b) show the posets $\left(\mathcal{D}_3^{(2)}, \leq_{\mathrm{rot}}\right)$ and $\left(\delta\left(\mathcal{D}_3^{(2)}\right), \leq_{\mathrm{dom}}\right)$.

REMARK 2.4.20

The converse of Lemma 2.4.19 is not true, i.e. the inverse map δ^{-1} is not order-preserving from $\left(\delta(\mathcal{D}_n^{(m)}),\leq_{\mathrm{dom}}\right)$ to $\left(\mathcal{D}_n^{(m)},\leq_{\mathrm{rot}}\right)$.

Consider for instance the Dyck paths $\mathfrak{q},\mathfrak{q}'\in\mathcal{D}_3$ with step sequences $\mathbf{u}_{\mathfrak{q}}=(0,1,1)$ and $\mathbf{u}_{\mathfrak{q}'}=(0,0,1)$. In view of Figure 19, we find that $\mathfrak{p}=\delta^{-1}(\mathfrak{q},\mathfrak{q})$ has step sequence $\mathbf{u}_{\mathfrak{p}}=(0,2,2)$, and $\mathfrak{p}'=\delta^{-1}(\mathfrak{q},\mathfrak{q}')$ has step sequence $\mathbf{u}_{\mathfrak{p}'}=(0,1,2)$. We have $\mathfrak{q}\leq_{\mathrm{dom}}\mathfrak{q}$ and $\mathfrak{q}\leq_{\mathrm{dom}}\mathfrak{q}'$, but $\mathfrak{p}\not\leq_{\mathrm{rot}}\mathfrak{p}'$.

REMARK 2.4.21

We can also consider δ as a map from $\left(\mathcal{D}_n^{(m)},\leq_{\mathrm{rot}}\right)$ to $\left(\delta(\mathcal{D}_n^{(m)}),\leq_{\mathrm{rot}}\right)$. However, in this case δ is not order-preserving.

Consider for instance $\mathfrak{p},\mathfrak{p}'\in\mathcal{D}_3^{(2)}$ with $\mathbf{u}_{\mathfrak{p}}=(0,1,2)$ and $\mathbf{u}_{\mathfrak{p}'}=(0,0,1)$. Then we have $\mathfrak{p}\leq_{\mathrm{rot}}\mathfrak{p}'$ and $\delta(\mathfrak{p})=(\mathfrak{q}_1,\mathfrak{q}_2)$ with $\mathbf{u}_{\mathfrak{q}_1}=(0,1,1),\mathbf{u}_{\mathfrak{q}_2}=(0,0,1)$, as well as $\delta(\mathfrak{p}')=(\mathfrak{q}_1',\mathfrak{q}_2')$ with $\mathbf{u}_{\mathfrak{q}_1'}=(0,0,1),\mathbf{u}_{\mathfrak{q}_2'}=(0,0,0)$. However, we have $\mathfrak{q}_1\not\leq_{\mathrm{rot}}\mathfrak{q}_1'$. See Figure 21(c) for an illustration of $\left(\delta(\mathcal{D}_3^{(2)}),\leq_{\mathrm{rot}}\right)$.

2.4.3. The m-Tamari Lattices as a Lattice Completion. In this section, we will finally investigate the m-cover poset of $\mathcal{T}_n$. We start with a simple observation on its cardinality.

PROPOSITION 2.4.22

For $m,n>0$, we have

$$\left|\mathcal{D}_n^{\langle m\rangle}\right|=\frac{n-1}{2}\Big(Cat(n)-2\Big)\binom{m}{2}+m\cdot Cat(n)-m+1.$$

PROOF. It is well known that $\left|\mathcal{D}_n\right|=\mathrm{Cat}(n)$, and it follows by construction that $\mathcal{T}_n$ has $n-1$ atoms. Moreover, [58, Theorem 5.3] states that the number of cover relations in $\mathcal{T}_n$ is precisely $\frac{n-1}{2}\cdot\mathrm{Cat}(n)$. Now plugging these values in Proposition 2.4.4 yields the result. □

We observe that $\left|\mathcal{D}_n^{\langle m\rangle}\right|<\mathrm{Cat}^{(m)}(n)$ for $n>3$ and $m>1$. Moreover, in these cases, Proposition 2.4.8 implies that $\mathcal{T}_n^{\langle m\rangle}$ is no longer a lattice. Since we want to use $\mathcal{T}_n^{\langle m\rangle}$ to realize $\mathcal{T}_n^{(m)}$, we need to consider a lattice completion of $\mathcal{T}_n^{\langle m\rangle}$. Recall that the *Dedekind-MacNeille completion of a poset $\mathcal{P}$*, denoted by $\mathbf{DM}(\mathcal{P})$, is the smallest lattice that contains $\mathcal{P}$ as a subposet. And indeed, it turns out that this lattice completion will do the trick. We need the following, well-known result.

THEOREM 2.4.23 ([9, Korollar 3])

For every finite lattice $\mathcal{L}$, we have

$$\mathcal{L}\cong\mathbf{DM}\big(\mathcal{J}(\mathcal{L})\cup\mathcal{M}(\mathcal{L})\big).$$

The main result of this section is the following.

Theorem 2.4.24

For $m,n>0$, we have $\mathcal{T}_n^{(m)}\cong\mathbf{DM}\big(\mathcal{T}_n^{\langle m\rangle}\big)$.

In order to apply Theorem 2.4.23, we need to understand the irreducible elements of $\mathcal{T}_n^{(m)}$ a bit better. In general, the study of the irreducible elements helps for understanding and characterizing certain classes of lattices, like for instance distributive and locally distributive lattices, see [76]. But let us return to the m-Tamari lattices.

Proposition 2.4.25

Let $m,n>0$, and let $\mathfrak{p}\in\mathcal{D}_n^{(m)}$ with step sequence $\mathbf{u}_{\mathfrak{p}}=(u_1,u_2,\ldots,u_n)$. Then, $\mathfrak{p}\in\mathcal{J}\big(\mathcal{T}_n^{(m)}\big)$ if and only if

$$u_j=\begin{cases}m(j-1), & \text{for } j\notin\{i,i+1,\ldots,k\},\\ m(j-i)-s, & \text{for } j\in\{i,i+1,\ldots,k\}\end{cases}\tag{2.22}$$

for exactly one $i\in\{1,2,\ldots,n\}$, where $k\in\{i+1,i+2,\ldots,n\}$ and $s\in\{1,2,\ldots,m\}$. Moreover, $\mathfrak{p}\in\mathcal{M}\big(\mathcal{T}_n^{(m)}\big)$ if and only if

$$u_j=\begin{cases}0, & \text{for } j\leq i,\\ s, & \text{for } j>i\end{cases}\tag{2.23}$$

where $s\in\{1,2,\ldots,mi\}$ and $i\in\{1,2,\ldots,n-1\}$.

Proof. Let $\mathfrak{p}\in\mathcal{D}_n^{(m)}$ with associated step sequence $\mathbf{u}_{\mathfrak{p}}=(u_1,u_2,\ldots,u_n)$.

First we focus on the join-irreducible elements of $\mathcal{T}_n^{(m)}$. Let $\mathfrak{p}\in\mathcal{D}_n^{(m)}$ have a step sequence of the form (2.22). Since the entries u_j for $j\notin\{i,i+1,\ldots,k\}$ are maximal, a lower cover of $\mathfrak{p}$ can only be obtained by increasing some of the values $u_i,u_{i+1},\ldots,u_k$. First, we increase only one entry, i.e. we consider the path $\mathfrak{p}_l\in\mathcal{D}_n^{(m)}$ given by the step sequence

$$\mathbf{u}_{\mathfrak{p}_l}=(u_1,u_2,\ldots,u_{l-1},u_l+1,u_{l+1},u_{l+2},\ldots,u_n),$$

where $l\in\{i,i+1,\ldots,k\}$. We show that $\mathfrak{p}_l\lessdot_{\mathrm{rot}}\mathfrak{p}$ only if $l=k$. Indeed, if $l<k$, then we have

$$u_{l+1}-(u_l+1)=ml-s-m(l-1)+s-1=m-1<m.$$

Hence u_{l+1} is contained in the primitive subsequence of $\mathfrak{p}_l$ at position l, which implies that $\mathfrak{p}_l$ is no lower cover of $\mathfrak{p}$. If $l=k$, then we have

$$u_{l+1}-(u_l+1)=ml-m(l-1)+s-1=m+s-1\geq m,$$

and hence u_{l+1} is not contained in the primitive subsequence of $\mathfrak{p}_l$ at position l. Thus $\mathfrak{p}_k\lessdot_{\mathrm{rot}}\mathfrak{p}$.

Now we increase at least two entries: for $l_1,l_2\in\{i,i+1,\ldots,k\}$ with $l_1<l_2$, we consider the path $\mathfrak{p}_{l_1,l_2}$ given by the step sequence

$$\mathbf{u}_{\mathfrak{p}_{l_1,l_2}}=(u_1,u_2,\ldots,u_{l_1-1},u_{l_1}+1,u_{l_1+1}+1,\ldots,u_{l_2}+1,u_{l_2+1},u_{l_2+2},\ldots,u_n).$$

As before we can show that $l_2=k$. Moreover, we have

$$u_{l_2}+1-(u_{l_1}+1)=u_{l_2}-u_{l_1}=m(l_2-l_1)\geq m.$$

Thus $\mathfrak{p}_{l_1,l_2}\leq_{\mathrm{rot}}\mathfrak{p}$, but $\mathfrak{p}_{l_1,l_2}$ is no lower cover of $\mathfrak{p}$. Again we see that $\mathfrak{p}_{l_1,l_2-1}\not\leq_{\mathrm{rot}}\mathfrak{p}$, which implies that every chain from $\mathfrak{p}_{l_1,l_2}$ to $\mathfrak{p}$ has to pass through $\mathfrak{p}_k$. Hence $\mathfrak{p}_k$ is the unique lower cover of $\mathfrak{p}$, and it follows that $\mathfrak{p}\in\mathcal{J}\big(\mathcal{T}_n^{(m)}\big)$ as desired.

For the converse, suppose that $\mathfrak{p}$ is not of the form (2.22). Then, we can find two indices j_1, j_2 such that $u_{j_1} = m(j_1-1)-s_1$ and $u_{j_2} = m(j_2-1)-s_2$ with $s_1, s_2 > 0$, and $s_1 \neq s_2$. Without loss of generality, we can assume that $j_2 = j_1+1$, which implies

$$u_{j_2} - u_{j_1} = mj_1 - s_2 - m(j_1-1) + s_1 = m + s_1 - s_2.$$

If $s_1 < s_2$, then we have $u_{j_2} - u_{j_1} < m$, and u_{j_2} lies in the primitive subsequence of $\mathbf{u}_{\mathfrak{p}}$ at position j_1. Now if we increase the entries of the primitive subsequence of $\mathbf{u}_{\mathfrak{p}}$ at position j_1 by one, then we obtain an m-Dyck path $\mathfrak{p}_1$, and the j_2-nd entry of $\mathbf{u}_{\mathfrak{p}_1}$ is contained in the primitive subsequence of $\mathbf{u}_{\mathfrak{p}_1}$ at position j_1. Then, $\mathfrak{p}_1 \lessdot_{\mathrm{rot}} \mathfrak{p}$. Analogously, if we increase the entries of the primitive subsequence of $\mathbf{u}_{\mathfrak{p}}$ at position j_2, then we obtain another m-Dyck path $\mathfrak{p}_2$ with $\mathfrak{p}_2 \lessdot_{\mathrm{rot}} \mathfrak{p}$. Clearly we have $\mathfrak{p}_1 \neq \mathfrak{p}_2$, which implies that $\mathfrak{p} \notin \mathcal{J}\big(\mathcal{T}_n^{(m)}\big)$. If $s_1 > s_2$, then $u_{j_2} - u_{j_1} > m$. Now increasing u_{j_1} by one yields an m-Dyck path $\mathfrak{p}_1$ with $\mathfrak{p}_1 \lessdot_{\mathrm{rot}} \mathfrak{p}$, and increasing the entries of the primitive subsequence of $\mathbf{u}_{\mathfrak{p}}$ by one yields an m-Dyck path $\mathfrak{p}_2$ with $\mathfrak{p}_2 \lessdot_{\mathrm{rot}} \mathfrak{p}$. Again we have $\mathfrak{p}_1 \neq \mathfrak{p}_2$, which implies $\mathfrak{p} \in \mathcal{J}\big(\mathcal{T}_n^{(m)}\big)$.

Now we focus on the meet-irreducible elements of $\mathcal{T}_n^{(m)}$. It follows immediately from the definition that the number of upper covers of $\mathfrak{p}$ is precisely the cardinality of the set $\big\{i \in \{1,2,\ldots,n-1\} \mid u_i < u_{i+1}\big\}$. Hence $\mathfrak{p} \in \mathcal{M}\big(\mathcal{T}_n^{(m)}\big)$ if and only if $\mathbf{u}_{\mathfrak{p}}$ is of the form (2.23). $\square$

In view of Proposition 2.4.25, we can easily compute the cardinality of the sets of irreducibles of $\mathcal{T}_n^{(m)}$.

Corollary 2.4.26

For $m,n > 0$ we have $\big|\mathcal{J}\big(\mathcal{T}_n^{(m)}\big)\big| = m\binom{n}{2} = \big|\mathcal{M}\big(\mathcal{T}_n^{(m)}\big)\big|$.

Proof. This is a straightforward computation. $\square$

Corollary 2.4.27

For $m,n > 0$ the lattice $\mathcal{T}_n^{(m)}$ is extremal.

Proof. This follows from Corollaries 2.3.3 and 2.4.26. $\square$

It was already observed in [76, Theorem 22] that $\mathcal{T}_n$ is an extremal lattice. However, Markowsky also observed in [76] that extremal lattices may contain intervals which are not extremal. Hence the extremality of $\mathcal{T}_n^{(m)}$ is not automatically implied by the fact that $\mathcal{T}_n^{(m)}$ is an interval in $\mathcal{T}_{mn}$.

Together with Proposition 2.4.6, Proposition 2.4.25 implies what the join- and the meet-irreducible elements of $\mathcal{T}_n^{(m)}$ and $\mathcal{T}_n^{\langle m\rangle}$ look like. Using the strip-decomposition, we can now show that these sets are actually isomorphic. In what follows, let $\mathfrak{o}$ denote the least element of $\mathcal{T}_n$, i.e. the Dyck path whose step sequence is $\mathbf{u}_{\mathfrak{o}} = (0,1,\ldots,n-1)$.

Proposition 2.4.28

For $m,n > 0$ the posets $\Big(\mathcal{J}\big(\mathcal{T}_n^{(m)}\big), \leq_{\mathrm{rot}}\Big)$ *and* $\Big(\mathcal{J}\big(\mathcal{T}_n^{\langle m\rangle}\big), \leq_{\mathrm{rot}}\Big)$ *are isomorphic.*

PROOF. It follows from Proposition 2.4.6 that

$$\mathcal{J}\big(\mathcal{T}_n^{\langle m\rangle}\big) = \Big\{\big(\mathfrak{o}^l, \mathfrak{q}^{m-l}\big) \mid \mathfrak{q} \in \mathcal{J}(\mathcal{T}_n) \text{ and } 0 \leq l < m\Big\}.$$

We show now that the strip-decomposition is a poset-isomorphism from $\Big(\mathcal{J}\big(\mathcal{T}_n^{(m)}\big), \leq_{\mathrm{rot}}\Big)$ to $\Big(\mathcal{J}\big(\mathcal{T}_n^{\langle m\rangle}\big), \leq_{\mathrm{rot}}\Big)$. First we show that $\delta\Big(\mathcal{J}\big(\mathcal{T}_n^{(m)}\big)\Big) \subseteq \mathcal{J}\big(\mathcal{T}_n^{\langle m\rangle}\big)$. Let $\mathfrak{p} \in \mathcal{J}\big(\mathcal{T}_n^{(m)}\big)$ with step sequence $\mathbf{u}_{\mathfrak{p}} = (u_1, u_2, \ldots, u_n)$. Proposition 2.4.25 implies then that

$$u_j = \begin{cases} m(j-1), & \text{for } j \notin \{i, i+1, \ldots, k\} \\ m(j-1) - s, & \text{for } j \in \{i, i+1, \ldots, k\}, \end{cases}$$

for exactly one $i \in \{1, 2, \ldots, n\}$ and some $k \in \{i+1, i+2, \ldots, n\}$, as well as some (fixed) $s \in \{1, 2, \ldots, m\}$. Now let $\delta(\mathfrak{p}) = (\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_m)$. It is straightforward to show that

$$\mathbf{u}_{\mathfrak{q}_j} = \begin{cases} (0, 1, \ldots, n-1), & \text{if } j \leq m-s, \\ (0, 1, \ldots, i-2, i-2, i-1, \ldots, k-2, k, k+1, \ldots, n-1), & \text{if } j > m-s. \end{cases} \tag{2.24}$$

Using Proposition 2.4.25 again, we see that $\delta(\mathfrak{p}) = (\mathfrak{o}, \mathfrak{o}, \ldots, \mathfrak{o}, \mathfrak{q}, \mathfrak{q}, \ldots, \mathfrak{q})$ for some $\mathfrak{q} \in \mathcal{J}(\mathcal{T}_n)$, and hence that $\delta(\mathfrak{p}) \in \mathcal{J}\big(\mathcal{T}_n^{\langle m\rangle}\big)$. Lemma 2.4.14 states that δ is injective, and Proposition 2.4.6 and Corollary 2.4.26 imply

$$\Big|\mathcal{J}\big(\mathcal{T}_n^{\langle m\rangle}\big)\Big| = m\binom{n}{2} = \Big|\mathcal{J}\big(\mathcal{T}_n^{(m)}\big)\Big|,$$

and it follows that $\delta\Big(\mathcal{J}\big(\mathcal{T}_n^{(m)}\big)\Big) = \mathcal{J}\big(\mathcal{T}_n^{\langle m\rangle}\big)$.

It remains to show that (in the given setting) δ and its inverse are both order-preserving. Let $\mathfrak{p}, \mathfrak{p}' \in \mathcal{J}\big(\mathcal{T}_n^{(m)}\big)$ and first assume that $\mathfrak{p} \leq_{\mathrm{rot}} \mathfrak{p}'$. Let i, k, s denote the parameters of $\mathfrak{p}$ according to Proposition 2.4.25, and let i', k', s' denote the analogous parameters of $\mathfrak{p}'$. Moreover, let $\mathbf{u}_{\mathfrak{p}} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{u}_{\mathfrak{p}'} = (u'_1, u'_2, \ldots, u'_n)$ denote the step sequences of $\mathfrak{p}$ and $\mathfrak{p}'$, respectively. In view of (2.24), we can write $\delta(\mathfrak{p}) = \big(\mathfrak{o}^{m-s}, \mathfrak{q}^s\big)$ and $\delta(\mathfrak{p}') = \big(\mathfrak{o}^{m-s'}, \mathfrak{q}'^{s'}\big)$ for $\mathfrak{q}, \mathfrak{q}' \in \mathcal{J}(\mathcal{T}_n)$. We need to show that $\mathfrak{q} \leq_{\mathrm{rot}} \mathfrak{q}'$ and $s \leq s'$. First of all, Lemma 2.2.6 implies that $i' \leq i$ and $k \leq k'$. For each $j \in \{i, i+1, \ldots, k\}$, we have $u_j = m(j-1) - s$ and $u'_j = m(j-1) - s'$, and again Lemma 2.2.6 implies $u_j \geq u'_j$, which yields $s \leq s'$. In view of (2.24), we find $\mathbf{u}_{\mathfrak{q}} = (0, 1, \ldots, i-2, i-2, i-1, \ldots, k-2, k, k+1, \ldots, n\}$ and $\mathbf{u}_{\mathfrak{q}'} = (0, 1, \ldots, i'-2, i'-2, i'-1, \ldots, k'-2, k', k'+1, \ldots, n\}$, and it is immediate that we can construct a sequence of cover relations in $\mathcal{T}_n$ that yields a chain from $\mathfrak{q}$ to $\mathfrak{q}'$. Hence $\mathfrak{q} \leq_{\mathrm{rot}} \mathfrak{q}'$ and thus $\delta(\mathfrak{p}) \leq_{\mathrm{rot}} \delta(\mathfrak{p}')$.

For the converse, suppose that $\big(\mathfrak{o}^{m-s}, \mathfrak{q}^s\big) \leq_{\mathrm{rot}} \big(\mathfrak{o}^{m-s'}, \mathfrak{q}'^{s'}\big)$. This implies immediately that $s \leq s'$ and $\mathfrak{q} \leq_{\mathrm{rot}} \mathfrak{q}'$. Since $\mathfrak{q}, \mathfrak{q}' \in \mathcal{J}(\mathcal{T}_n)$, we can apply Proposition 2.4.25, and we obtain two parameters i, k for $\mathfrak{q}$ and i', k' for $\mathfrak{q}'$ which satisfy $i' \leq i \leq k \leq k'$. Thus we have $u_j = m(j-1)$ for $j \notin \{i, i+1, \ldots, k\}$ and $u_j = m(j-1) - s$ for $j \in \{i, i+1, \ldots, k\}$, as well as $u'_j = m(j-1)$ for $j \notin \{i', i'+1, \ldots, k'\}$ and $u'_j = m(j-1) - s'$ for $j' \in \{i', i'+1, \ldots, k'\}$. Again it is immediate that we can construct a sequence of cover relations in $\mathcal{T}_n^{(m)}$ that yields a chain from $\mathfrak{p}$ to $\mathfrak{p}'$. Hence $\mathfrak{p} \leq_{\mathrm{rot}} \mathfrak{p}'$, and we are done. $\square$

PROPOSITION 2.4.29

For $m, n > 0$ the posets $\Big(\mathcal{M}\big(\mathcal{T}_n^{(m)}\big), \leq_{\mathrm{rot}}\Big)$ and $\Big(\mathcal{M}\big(\mathcal{T}_n^{\langle m\rangle}\big), \leq_{\mathrm{rot}}\Big)$ are isomorphic.

Proof. If $n \leq 2$, then it follows that $\mathcal{T}_n$ is either a singleton or a 2-chain, and hence $\mathcal{T}_n^{(m)} \cong \mathcal{T}_n^{\langle m\rangle}$. So suppose that $n > 2$. Since $\mathcal{T}_n$ has $n-1$ atoms and $n-1$ coatoms, Proposition 2.4.6 implies that

$$\mathcal{M}\big(\mathcal{T}_n^{\langle m\rangle}\big) = \Big\{\big(\mathfrak{q}^l, (\mathfrak{q}^\star)^{m-l}\big) \mid \mathfrak{q} \in \mathcal{M}(\mathcal{T}_n) \text{ and } 0 < l \leq m\Big\}.$$

We show now that the strip-decomposition is a poset isomorphism from $\Big(\mathcal{M}\big(\mathcal{T}_n^{(m)}\big), \leq_{\mathrm{rot}}\Big)$ to $\Big(\mathcal{M}\big(\mathcal{T}_n^{\langle m\rangle}\big), \leq_{\mathrm{rot}}\Big)$. First we show that $\delta\Big(\mathcal{M}\big(\mathcal{T}_n^{(m)}\big)\Big) \subseteq \mathcal{M}\big(\mathcal{T}_n^{\langle m\rangle}\big)$. If $\mathfrak{p} \in \mathcal{M}\big(\mathcal{T}_n^{(m)}\big)$, then Proposition 2.4.25 implies that $\mathbf{u}_{\mathfrak{p}} = (0,0,\ldots,a,a,\ldots,a)$, where the a appears first in the $(i+1)$-st position and satisfies $1 \leq a \leq mi$. If we write $a = mk+t$ with $t \in \{0,1,\ldots,m-1\}$ and $k \in \{0,1,\ldots,i\}$, then we obtain $\delta(\mathfrak{p}) = (\mathfrak{q}_1,\mathfrak{q}_2,\ldots,\mathfrak{q}_m)$, where

$$\mathbf{u}_{\mathfrak{q}_j} = \begin{cases}(0,0,\ldots,0,k+1,k+1,\ldots,k+1), & \text{if } j \leq t,\\ (0,0,\ldots,0,k,k,\ldots,k), & \text{if } j > t\end{cases}, \tag{2.25}$$

and $k+1$ respectively k first appears in the $(i+1)$-st position of $\mathbf{u}_{\mathfrak{q}_j}$. It follows from Proposition 2.4.25 that $\mathfrak{q}_j \in \mathcal{M}(\mathcal{T}_n)$, and by definition of the rotation order, we have $\mathfrak{q}_t \lessdot_{\mathrm{rot}} \mathfrak{q}_{t+1}$. Hence $\delta(\mathfrak{p}) \in \mathcal{M}\big(\mathcal{T}_n^{\langle m\rangle}\big)$. Lemma 2.4.14 states that δ is injective, and Proposition 2.4.6 and Corollary 2.4.26 imply

$$\Big|\mathcal{M}\big(\mathcal{T}_n^{\langle m\rangle}\big)\Big| = m\binom{n}{2} = \Big|\mathcal{M}\big(\mathcal{T}_n^{(m)}\big)\Big|,$$

and it follows that $\delta\Big(\mathcal{M}\big(\mathcal{T}_n^{(m)}\big)\Big) = \mathcal{M}\big(\mathcal{T}_n^{\langle m\rangle}\big)$.

It remains to show that (in the given setting) δ and its inverse are both order-preserving. Let $\mathfrak{p},\mathfrak{p}' \in \mathcal{M}\big(\mathcal{T}_n^{(m)}\big)$, and first assume that $\mathfrak{p} \leq_{\mathrm{rot}} \mathfrak{p}'$. By assumption, we can write $\mathbf{u}_{\mathfrak{p}} = (0,0,\ldots,0,a,a,\ldots,a)$ with $a = mk+t$ and $\mathbf{u}_{\mathfrak{p}'} = (0,0,\ldots,0,a',a',\ldots,a')$ with $a' = mk'+t'$, where the nonzero entries appear first in the i-th and i'-th position, respectively. Since $\mathfrak{p} \leq_{\mathrm{rot}} \mathfrak{p}'$, we conclude that $i = i'$, and either $k' < k$ or $k' = k$ and $t' < t$. Let $\delta(\mathfrak{p}) = (\mathfrak{q}_1,\mathfrak{q}_2,\ldots,\mathfrak{q}_m)$, and let $\delta(\mathfrak{p}') = (\mathfrak{q}'_1,\mathfrak{q}'_2,\ldots,\mathfrak{q}'_m)$.
(i) If $t' < t$, then $k' \leq k$, and in view of (2.25), we obtain

$$\begin{aligned}\mathbf{u}_{\mathfrak{q}_1} = \mathbf{u}_{\mathfrak{q}_2} = \cdots = \mathbf{u}_{\mathfrak{q}_{t'}} &= (0,0,\ldots,k+1,k+1,\ldots,k+1),\\ \mathbf{u}_{\mathfrak{q}_{t'+1}} = \mathbf{u}_{\mathfrak{q}_{t'+2}} = \cdots = \mathbf{u}_{\mathfrak{q}_t} &= (0,0,\ldots,0,k+1,k+1,\ldots,k+1), \quad \text{and}\\ \mathbf{u}_{\mathfrak{q}_{t+1}} = \mathbf{u}_{\mathfrak{q}_{t+2}} = \cdots = \mathbf{u}_{\mathfrak{q}_m} &= (0,0,\ldots,0,k,k,\ldots,k),\end{aligned}$$

as well as

$$\begin{aligned}\mathbf{u}_{\mathfrak{q}'_1} = \mathbf{u}_{\mathfrak{q}'_2} = \cdots = \mathbf{u}_{\mathfrak{q}'_{t'}} &= (0,0,\ldots,k'+1,k'+1,\ldots,k'+1),\\ \mathbf{u}_{\mathfrak{q}'_{t'+1}} = \mathbf{u}_{\mathfrak{q}'_{t'+2}} = \cdots = \mathbf{u}_{\mathfrak{q}'_t} &= (0,0,\ldots,0,k',k',\ldots,k'), \quad \text{and}\\ \mathbf{u}_{\mathfrak{q}'_{t+1}} = \mathbf{u}_{\mathfrak{q}'_{t+2}} = \cdots = \mathbf{u}_{\mathfrak{q}'_m} &= (0,0,\ldots,0,k',k',\ldots,k'),\end{aligned}$$

and it follows immediately that $\mathfrak{q}_j \leq_{\mathrm{rot}} \mathfrak{q}'_j$ for all $j \in \{1,2,\ldots,m\}$.
(ii) If $t' \geq t$, then $k' < k$, and in view of (2.25), we obtain

$$\begin{aligned}\mathbf{u}_{\mathfrak{q}_1} = \mathbf{u}_{\mathfrak{q}_2} = \cdots = \mathbf{u}_{\mathfrak{q}_t} &= (0,0,\ldots,k+1,k+1,\ldots,k+1),\\ \mathbf{u}_{\mathfrak{q}_{t+1}} = \mathbf{u}_{\mathfrak{q}_{t+2}} = \cdots = \mathbf{u}_{\mathfrak{q}_{t'}} &= (0,0,\ldots,0,k,k,\ldots,k), \quad \text{and}\\ \mathbf{u}_{\mathfrak{q}_{t'+1}} = \mathbf{u}_{\mathfrak{q}_{t'+2}} = \cdots = \mathbf{u}_{\mathfrak{q}_m} &= (0,0,\ldots,0,k,k,\ldots,k),\end{aligned}$$

as well as

$$\begin{aligned}
\mathbf{u}_{\mathfrak{q}'_1} &= \mathbf{u}_{\mathfrak{q}'_2} = \cdots = \mathbf{u}_{\mathfrak{q}'_t} = (0,0,\ldots,k'+1,k'+1,\ldots,k'+1),\\
\mathbf{u}_{\mathfrak{q}'_{t+1}} &= \mathbf{u}_{\mathfrak{q}'_{t+2}} = \cdots = \mathbf{u}_{\mathfrak{q}'_{t'}} = (0,0,\ldots,0,k'+1,k'+1,\ldots,k'+1), \quad \text{and}\\
\mathbf{u}_{\mathfrak{q}'_{t'+1}} &= \mathbf{u}_{\mathfrak{q}'_{t'+2}} = \cdots = \mathbf{u}_{\mathfrak{q}'_m} = (0,0,\ldots,0,k',k',\ldots,k'),
\end{aligned}$$

and it follows again that $\mathfrak{q}_j \leq_{\mathrm{rot}} \mathfrak{q}'_j$ for all $j \in \{1,2,\ldots,m\}$. Hence we have $\delta(\mathfrak{p}) \leq_{\mathrm{rot}} \delta(\mathfrak{p}')$. We can show analogously, that if $\delta(\mathfrak{p}) \leq_{\mathrm{rot}} \delta(\mathfrak{p}')$, then $\mathfrak{p} \leq_{\mathrm{rot}} \mathfrak{p}'$ and we are done. $\square$

The next result states that the irreducible elements of $\mathcal{T}_n^{\langle m\rangle}$ are actually enough to construct the Dedekind-MacNeille completion.

Proposition 2.4.30

Every element in $\mathcal{D}_n^{\langle m\rangle} \setminus \{(\mathfrak{o}^m)\}$ can be expressed as a componentwise join of elements in $\mathcal{J}(\mathcal{T}_n^{\langle m\rangle})$.

Proof. Let $(\mathfrak{o}^{l_0}, \mathfrak{q}^{l_1}, (\mathfrak{q}')^{l_2}) \in \mathcal{D}_n^{\langle m\rangle}$. Recall for instance from [97, Theorem 8.1] that every element of $\mathcal{T}_n$ has a so-called *canonical join-representation*, i.e. that every element of $\mathcal{T}_n$ can be expressed as the join of a unique, minimal set of join-irreducible elements of $\mathcal{T}_n$. Now suppose that $\{\mathfrak{r}_1, \mathfrak{r}_2, \ldots, \mathfrak{r}_k\}$ is the canonical join-representation of $\mathfrak{q}$ and $\{\mathfrak{r}'_1, \mathfrak{r}'_2, \ldots, \mathfrak{r}'_{k'}\}$ is the canonical join-representation of $\mathfrak{q}'$, i.e. $\mathfrak{q} = \mathfrak{r}_1 \vee \mathfrak{r}_2 \vee \cdots \vee \mathfrak{r}_k$ and $\mathfrak{q}' = \mathfrak{r}'_1 \vee \mathfrak{r}'_2 \vee \cdots \vee \mathfrak{r}'_{k'}$ with $\mathfrak{r}_i, \mathfrak{r}'_{i'} \in \mathcal{J}(\mathcal{T}_n)$ for $i \in \{1,2,\ldots,k\}$ and $i' \in \{1,2,\ldots,k'\}$. Since $\mathfrak{q} \lessdot_{\mathrm{rot}} \mathfrak{q}'$, we can conclude that $k < k'$, and without loss of generality we can assume that $\mathfrak{r}_i = \mathfrak{r}'_i$ for $i \in \{1,2,\ldots,k\}$. Thus if we abbreviate $\mathbf{w}_i = (\mathfrak{o}^{l_0}, \mathfrak{r}_i^{l_1+l_2})$ for $i \in \{1,2,\ldots,k\}$, and $\mathbf{w}'_{i'} = (\mathfrak{o}^{l_0+l_1}, (\mathfrak{r}')^{l_2}_{k+i'})$ for $i' \in \{1,2,\ldots,k'-k\}$, then we have $\mathbf{w}_i, \mathbf{w}'_{i'} \in \mathcal{J}(\mathcal{T}_n^{\langle m\rangle})$ for $i \in \{1,2,\ldots,k\}$ and $i' \in \{1,2,\ldots,k'-k\}$. Moreover, we have

$$\begin{aligned}
\mathbf{w}_1 \vee \mathbf{w}_2 \vee \cdots \vee \mathbf{w}_k \vee \mathbf{w}'_1 \vee \mathbf{w}'_2 \vee \cdots \vee \mathbf{w}'_{k'-k} &= \left(\mathfrak{o}^{l_0}, \Big(\bigvee_{i=1}^{k} \mathfrak{r}_i\Big)^{l_1}, \Big(\bigvee_{i=1}^{k'} \mathfrak{r}'_i\Big)\right)\\
&= (\mathfrak{o}^{l_0}, \mathfrak{q}^{l_1}, (\mathfrak{q}')^{l_2}).
\end{aligned}$$

$\square$

Now everything is set to prove the connection between $\mathcal{T}_n^{(m)}$ and the m-cover poset of $\mathcal{T}_n$ stated in Theorem 2.4.24.

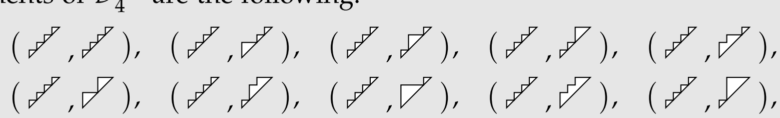
Proof of Theorem 2.4.24. By using Theorem 2.4.23 and Propositions 2.4.28–2.4.30, we obtain

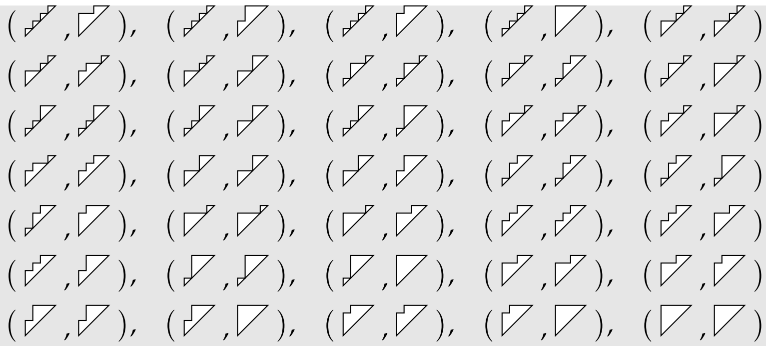
$$\mathcal{T}_n^{(m)} \cong \mathbf{DM}\Big(\mathcal{J}(\mathcal{T}_n^{(m)}) \cup \mathcal{M}(\mathcal{T}_n^{(m)})\Big) \cong \mathbf{DM}\Big(\mathcal{J}(\mathcal{T}_n^{\langle m\rangle}) \cup \mathcal{M}(\mathcal{T}_n^{\langle m\rangle})\Big) \cong \mathbf{DM}\Big(\mathcal{T}_n^{\langle m\rangle}\Big).$$

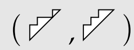
$\square$

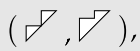
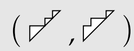
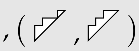
Example 2.4.31

Let us illustrate the case $m = 2$ and $n = 4$. The Tamari lattice $\mathcal{T}_4$ is shown again in Figure 22, and the elements of $\mathcal{D}_4^{\langle 2\rangle}$ are the following:

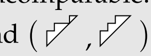
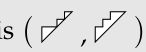


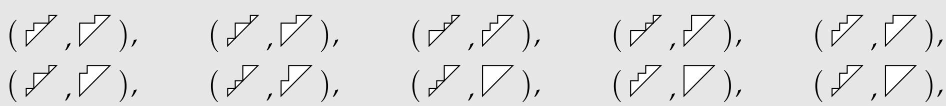


Now we notice that for instance the pairs  and  do not have a meet in $\mathcal{T}_4^{\langle 2\rangle}$, since

 $\leq_{\mathrm{rot}}$ , , and  $\leq_{\mathrm{rot}}$ , ,

but  and  are mutually incomparable.

The componentwise meet of  and  is . If we now successively add all the missing meets, then we can check that we have to include the following ten elements:



and these 55 elements form a lattice which is indeed isomorphic to $\mathcal{T}_4^{(2)}$, see Figure 23.

Theorem 2.4.24 states that we can realize $\mathcal{T}_n^{(m)}$ as a poset of m-tuples of Dyck paths equipped with componentwise rotation order. However, this realization is rather implicit, since we only know what the elements of $\mathcal{T}_n^{\langle m\rangle}$ look like, but we have no explicit description of the elements added during the lattice completion. We present now an explicit, but only conjectural description of this realization. Let $\wedge$ and $\vee$ denote meet and join in $\mathcal{T}_n$, and for $i,j\in\{1,2,\ldots,m\}$ consider the map

$$\beta_{i,j}:(\mathcal{D}_n)^m\to(\mathcal{D}_n)^m,\quad(\mathfrak{q}_1,\mathfrak{q}_2,\ldots,\mathfrak{q}_m)\mapsto$$
$$(\mathfrak{q}_1,\mathfrak{q}_2,\ldots,\mathfrak{q}_{i-1},\mathfrak{q}_i\wedge\mathfrak{q}_j,\mathfrak{q}_{i+1},\mathfrak{q}_{i+2},\ldots,\mathfrak{q}_{j-1},\mathfrak{q}_i\vee\mathfrak{q}_j,\mathfrak{q}_{j+1},\mathfrak{q}_{j+2},\ldots,\mathfrak{q}_m).$$

We call the composition

$$\beta=\beta_{m-1,m}\circ\beta_{m-2,m}\circ\cdots\circ\beta_{2,3}\circ\beta_{1,m}\circ\cdots\circ\beta_{1,3}\circ\beta_{1,2}\tag{2.26}$$

the *bouncing map*. (This composition acts from the left, i.e. we first apply $\beta_{1,2}$, then $\beta_{1,3}$, and so on.) In particular, it follows that $\beta(\mathfrak{q}_1,\mathfrak{q}_2,\ldots,\mathfrak{q}_m)$ is a multichain in $\mathcal{T}_n$. If we abbreviate $\zeta=\beta\circ\delta$ (again acting from the left), then we obtain a map

$$\zeta:\mathcal{D}_n^{(m)}\to(\mathcal{D}_n)^m,\quad\mathfrak{p}\mapsto\zeta(\mathfrak{p}).\tag{2.27}$$

Example 2.4.32

Let $\mathfrak{p}$ be the 5-Dyck path of length 36 with strip-decomposition $\delta(\mathfrak{p})=(\mathfrak{q}_1,\mathfrak{q}_2,\mathfrak{q}_3,\mathfrak{q}_4,\mathfrak{q}_5)$ as illustrated in Figure 18. There are only two non-trivial steps in the application of the

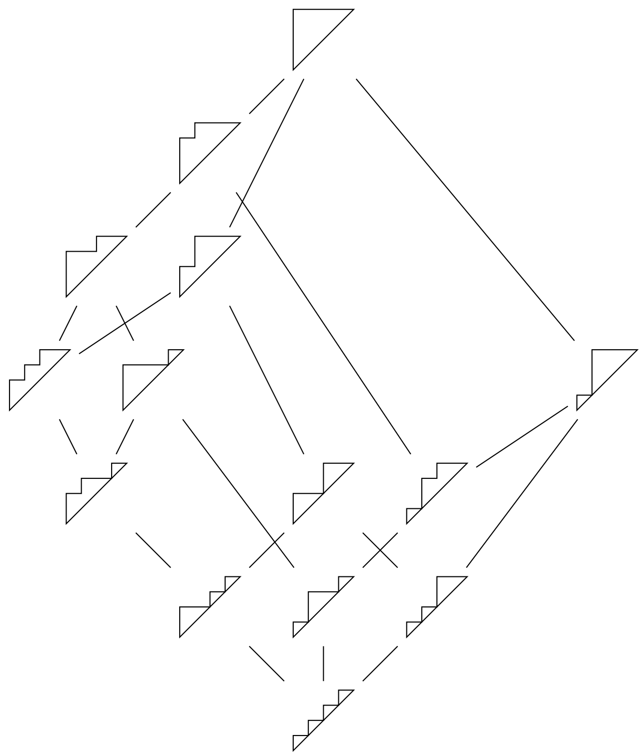


FIGURE 22. The Tamari lattice $\mathcal{T}_4$ again.

bouncing map:

$$i=1, j=3 : \mathfrak{q}_1 \wedge \mathfrak{q}_3 = \mathfrak{o}, \text{ and } \mathfrak{q}_1 \vee \mathfrak{q}_3 = \mathfrak{r}_1 \text{ with } \mathbf{u}_{\mathfrak{r}_1} = (0,0,0,1,2,5);$$
$$i=2, j=4 : \mathfrak{q}_2 \wedge \mathfrak{q}_4 = \mathfrak{r}_2 \text{ with } \mathbf{u}_{\mathfrak{r}_2} = (0,1,1,2,4,5), \text{ and } \mathfrak{q}_2 \vee \mathfrak{q}_4 = \mathfrak{q}_5.$$

Thus we have $\zeta(\mathfrak{p}) = (\mathfrak{o}, \mathfrak{r}_2, \mathfrak{r}_1, \mathfrak{q}_5, \mathfrak{q}_5)$.

Computer experiments suggest the following somewhat surprising property of ζ.

CONJECTURE 2.4.33

The posets $\left(\mathcal{D}_n^{(m)}, \leq_{\mathrm{rot}}\right)$ *and* $\left(\zeta(\mathcal{D}_n^{(m)}), \leq_{\mathrm{rot}}\right)$ *are isomorphic.*

REMARK 2.4.34

In general, β is not an order-preserving map from $\left(\delta(\mathcal{D}_n^{(m)}), \leq_{\mathrm{dom}}\right)$ to $\left(\zeta(\mathcal{D}_n^{(m)}), \leq_{\mathrm{rot}}\right)$.

Consider for instance the Dyck paths $\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_1', \mathfrak{q}_2' \in \mathcal{D}_5$ given by the height sequences $\mathbf{h}_{\mathfrak{q}_1} = (1,3,3,4,5), \mathbf{h}_{\mathfrak{q}_2} = (2,3,4,4,5), \mathbf{h}_{\mathfrak{q}_1'} = (2,3,3,5,5)$ and $\mathbf{h}_{\mathfrak{q}_2'} = (2,3,4,5,5)$. Then, it follows that $\mathfrak{q}_1 \leq_{\mathrm{dom}} \mathfrak{q}_1'$ and $\mathfrak{q}_2 \leq_{\mathrm{dom}} \mathfrak{q}_2'$. Moreover, the pairs $(\mathfrak{q}_1, \mathfrak{q}_2)$ and $(\mathfrak{q}_1', \mathfrak{q}_2')$ satisfy the conditions from Lemma 2.4.18, which implies that they are indeed contained in $\delta(\mathcal{D}_5^{(2)})$.

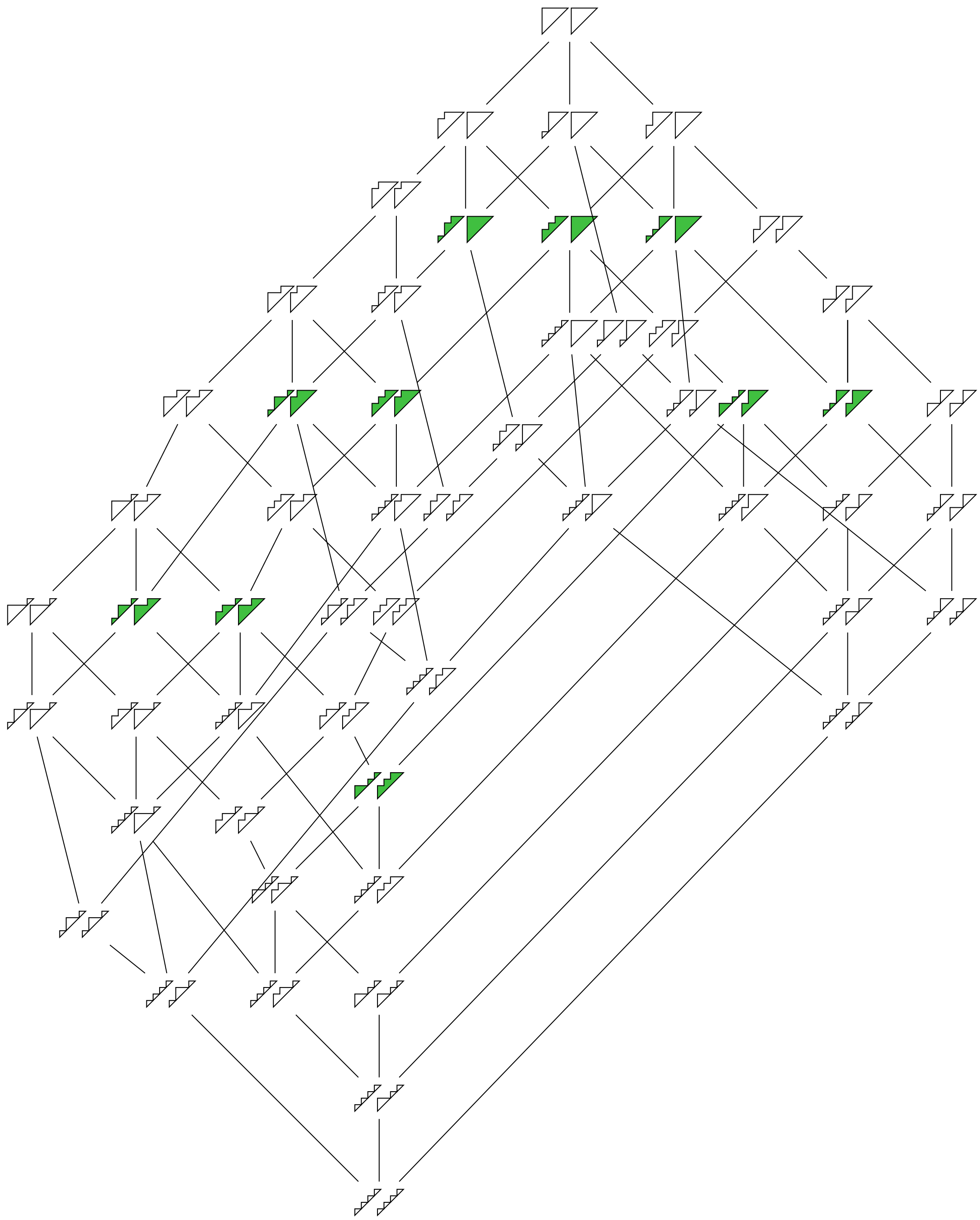


FIGURE 23. The lattice $\mathbf{DM}\big(\mathcal{T}_4^{\langle 2\rangle}\big)$. The highlighted elements are added during the lattice completion.

Further we have

$$\mathbf{h}_{\mathfrak{q}_1\wedge\mathfrak{q}_2} = (1,2,3,4,5) \text{ and } \mathbf{h}_{\mathfrak{q}_1\vee\mathfrak{q}_2} = (3,3,4,4,5), \quad \text{as well as}$$
$$\mathbf{h}_{\mathfrak{q}'_1\wedge\mathfrak{q}'_2} = (2,3,3,4,5) \text{ and } \mathbf{h}_{\mathfrak{q}'_1\vee\mathfrak{q}'_2} = (2,3,5,5,5).$$

The corresponding step sequences are

$$\mathbf{u}_{\mathfrak{q}_1\wedge\mathfrak{q}_2} = (0,1,2,3,4) \text{ and } \mathbf{u}_{\mathfrak{q}_1\vee\mathfrak{q}_2} = (0,0,0,2,4), \quad \text{as well as}$$
$$\mathbf{u}_{\mathfrak{q}'_1\wedge\mathfrak{q}'_2} = (0,0,1,3,4) \text{ and } \mathbf{u}_{\mathfrak{q}'_1\vee\mathfrak{q}'_2} = (0,0,1,2,2),$$

which implies $\beta(\mathfrak{q}_1,\mathfrak{q}_2) \not\leq_{\mathrm{rot}} \beta(\mathfrak{q}'_1,\mathfrak{q}'_2)$.

However, if $\mathfrak{p},\mathfrak{p}' \in \mathcal{D}_5^{(2)}$ are the 2-Dyck paths satisfying $\delta(\mathfrak{p}) = (\mathfrak{q}_1,\mathfrak{q}_2)$ and $\delta(\mathfrak{p}') = (\mathfrak{q}'_1,\mathfrak{q}'_2)$, then we can quickly check that $\mathbf{u}_{\mathfrak{p}} = (0,1,2,5,8)$ and $\mathbf{u}_{\mathfrak{p}'} = (0,0,2,5,6)$, which implies $\mathfrak{p} \not\leq_{\mathrm{rot}} \mathfrak{p}'$. So this is *not* a counterexample to Conjecture 2.4.33.

The following conjecture turns out to be equivalent to Conjecture 2.4.33, as we will explain afterwards.

CONJECTURE 2.4.35

The map ζ from (2.27) *is injective.*

REMARK 2.4.36

Suppose that Conjecture 2.4.35 is true, and let $\mathrm{Irr}(\mathcal{T}_n^{(m)}) = \mathcal{J}(\mathcal{T}_n^{(m)}) \cup \mathcal{M}(\mathcal{T}_n^{(m)})$. Then, it follows that $\left|\mathcal{D}_n^{(m)}\right| = \left|\zeta(\mathcal{D}_n^{(m)})\right|$, and we can quickly check that if $\mathfrak{p} \in \mathrm{Irr}(\mathcal{T}_n^{(m)})$, then $\zeta(\mathfrak{p}) = \delta(\mathfrak{p})$. Hence it follows from Theorem 2.4.23, and Propositions 2.4.28 and 2.4.29 that

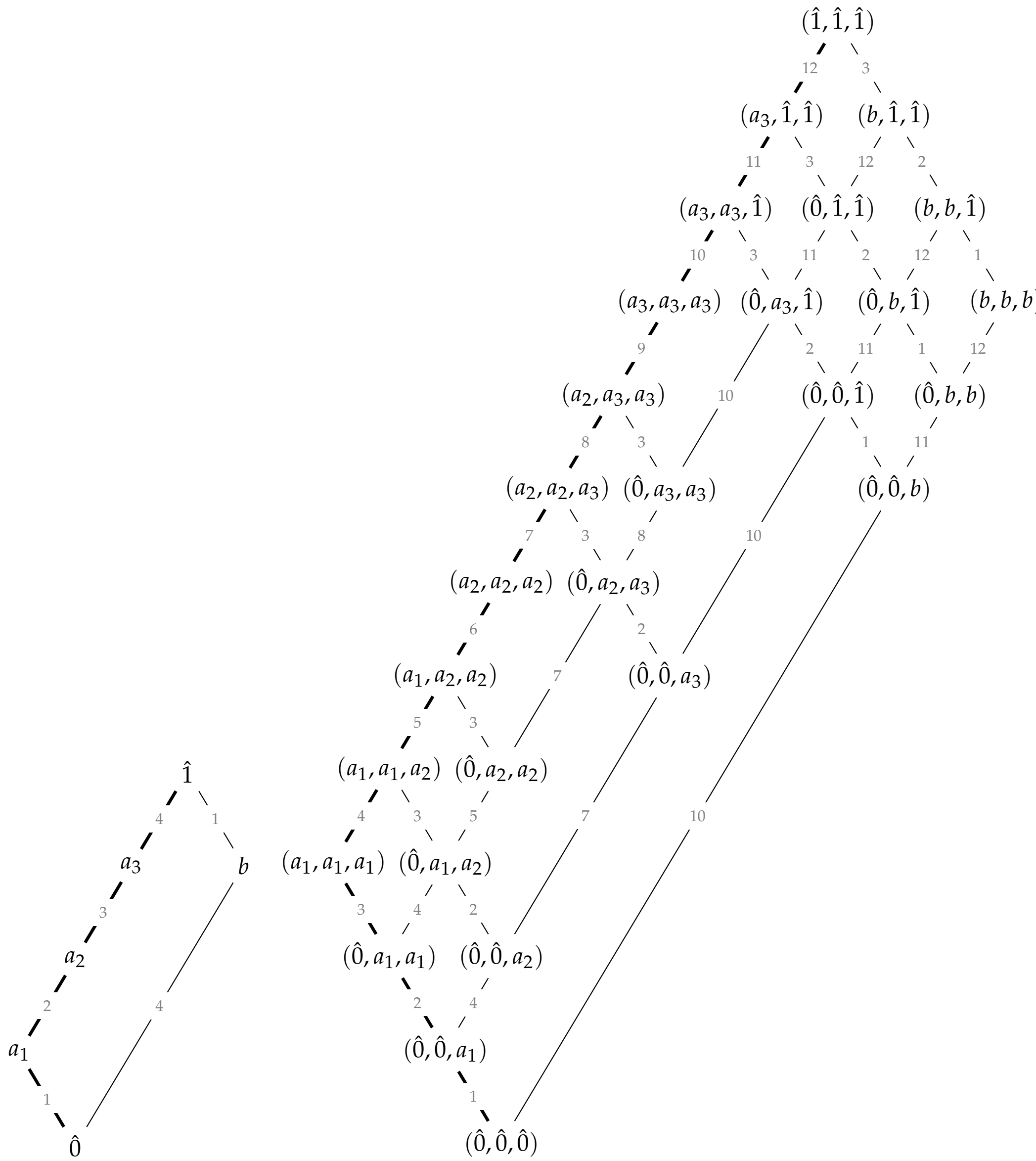
$$\begin{aligned}
\left(\mathcal{D}_n^{(m)}, \leq_{\mathrm{rot}}\right) &= \mathcal{T}_n^{(m)}\\
&\cong \mathbf{DM}\Big(\mathrm{Irr}\big(\mathcal{T}_n^{(m)}\big)\Big)\\
&\cong \mathbf{DM}\Big(\mathrm{Irr}\big(\delta(\mathcal{D}_n^{(m)})\big), \leq_{\mathrm{rot}}\big)\Big)\\
&\cong \mathbf{DM}\Big(\mathrm{Irr}\big(\zeta(\mathcal{D}_n^{(m)})\big), \leq_{\mathrm{rot}}\big)\Big)\\
&\supseteq \Big(\zeta\big(\mathcal{D}_n^{(m)}\big), \leq_{\mathrm{rot}}\Big).
\end{aligned}$$

Since ζ is injective, the claim of Conjecture 2.4.33 follows.

Conversely, if Conjecture 2.4.33 is true, then Conjecture 2.4.35 follows immediately, since poset isomorphisms are by definition injective.

2.4.4. A Family of "m-Tamari Like" Lattices for the Dihedral Groups. We complete this section by defining a family of lattices associated with the dihedral group $I_2(k)$ which is parametrized by a positive integer m. In particular let $\mathcal{C}_k$ denote the lattice consisting of $k+2$ elements whose proper part is the disjoint union of a $(k-1)$-chain and a singleton. The poset of interest in this section is $\mathcal{C}_k^{\langle m\rangle}$, the m-cover poset of $\mathcal{C}_k$. For later use, we denote the elements of the $(k-1)$-chain of $\mathcal{C}_k$ by $a_1, a_2, \ldots, a_{k-1}$, we denote the singleton element of $\mathcal{C}_k$ by b, and we denote the least and the greatest element of $\mathcal{C}_k$ by $\hat{0}$ and $\hat{1}$, respectively. See Figure 24 for an illustration.

We can use the results from Section 2.4.1 to determine certain properties of $\mathcal{C}_k^{\langle m\rangle}$.



(a) The lattice $\mathcal{C}_4$. (b) The lattice $\mathcal{C}_4^{\langle 3\rangle}$.

FIGURE 24. Two Fuß-Catalan lattices associated with the dihedral group $I_2(4)$. The highlighted chains consist of left-modular elements, and the edge-labeling is the one defined in (1.2).

PROPOSITION 2.4.37

For $k > 1$ and $m > 0$ the poset $\mathcal{C}_k^{\langle m\rangle}$ is a lattice. Its length is mk and its cardinality is $\binom{m+1}{2}k + m + 1$.

Proof. By definition, the Hasse diagram of $\mathcal{C}_k$ with $\hat{0}$ removed is a tree, hence the lattice property of $\mathcal{C}_k^{\langle m\rangle}$ follows from Corollary 2.4.10. Moreover, since $k > 1$, the lattice $\mathcal{C}_k^{\langle m\rangle}$ has length k, cardinality $k+2$, it has two atoms and $k+1$ cover relations. Hence the results on the length and the cardinality of $\mathcal{C}_k^{\langle m\rangle}$ follow from Proposition 2.4.4. $\square$

Disclaimer 2.4.38

By definition, the lattice $\mathcal{C}_1$ has three elements, and in this case, the m-cover poset $\mathcal{C}_1^{\langle m\rangle}$ has $\binom{m}{2}+2m+1$ elements, which does not agree with the formula from Proposition 2.4.37. Hence we will always exclude the case $k=1$ from our considerations.

In the beginning of this section we have claimed that the lattices $\mathcal{C}_k$ are associated with the dihedral group $I_2(k)$. Indeed, as we will see in Chapter 3 there exist generalizations of the Tamari lattices for all Coxeter groups, and thus in particular for the dihedral groups. The degrees of $I_2(k)$ are $d_1 = 2$ and $d_2 = k$, provided that $k > 1$. Hence according to (0.3), we can define Fuß-Catalan numbers for $I_2(k)$ by

$$\operatorname{Cat}^{(m)}\big(I_2(k)\big) = \frac{mk+2}{2}\cdot\frac{mk+k}{k} = \frac{m^2k+mk+2m+2}{2} = \binom{m+1}{2}k+m+1,$$

which is precisely the cardinality of $\mathcal{C}_k^{\langle m\rangle}$. Thus the lattices $\mathcal{C}_k^{\langle m\rangle}$ can be seen as an m-Tamari lattice of type I. The next lemma states that this connection is consistent with respect to the fact that the symmetric group A_2 is isomorphic to the dihedral group $I_2(3)$.

Lemma 2.4.39

For $m > 0$, we have $\mathcal{T}_3^{(m)} \cong \mathcal{C}_3^{\langle m\rangle}$.

Proof. We can see for instance from Figure 9 that $\mathcal{T}_3 \cong \mathcal{C}_3$. Proposition 2.4.8 implies that $\mathcal{T}_3^{\langle m\rangle}$ is a lattice for all $m > 0$. Hence in view of Theorem 2.4.24, we conclude

$$\mathcal{T}_3^{(m)} \cong \mathbf{DM}\big(\mathcal{T}_3^{\langle m\rangle}\big) \cong \mathcal{T}_3^{\langle m\rangle} \cong \mathcal{C}_3^{\langle m\rangle}.$$

$\square$

Now it is evident, to ask whether this construction also works for the other Coxeter groups. However, we notice that this already fails in type B. There are explicit descriptions of Tamari lattices of type B, see [93,122], which fit nicely in the framework of Cambrian lattices described in the next chapter. The corresponding Fuß-Catalan number of type B is

$$\operatorname{Cat}^{(m)}(B_n) = \prod_{i=1}^{n}\frac{2mn+2i}{2i} = \binom{(m+1)n}{n},$$

and for $n = 3$ and $m = 2$, we obtain $\operatorname{Cat}^{(2)}(B_3) = \binom{9}{3} = 84$. However, we can check that the corresponding type B Tamari lattice has 20 elements, 3 atoms and 30 cover relations. Hence Proposition 2.4.4 implies that its 2-cover poset has only 66 elements, and we can check that the corresponding Dedekind-MacNeille completion has 88 elements. See Figure 32 for an illustration of this lattice. See also Appendix A for some tables containing the cardinalities of the m-cover posets and of their Dedekind-MacNeille completion for all Cambrian lattices associated with finite Coxeter groups of rank at most 4 and $m \leq 4$.

We conclude this chapter with a short topological investigation of the lattices $\mathcal{C}_k^{\langle m\rangle}$. In particular, we show that the lattices $\mathcal{C}_k^{\langle m\rangle}$ are trim for all $k>1$ and $m>0$. See Section 1.1.5 for the terminology. Let $C_k^{\langle m\rangle}=\{\hat{0},a_1,a_2,\ldots,a_{k-1},b,\hat{1}\}$ denote the ground set of $\mathcal{C}_k^{\langle m\rangle}$, and let $\leq_k$ denote the partial order of $\mathcal{C}_k^{\langle m\rangle}$.

PROPOSITION 2.4.40

For $k>1$ and $m>0$, the lattice $\mathcal{C}_k^{\langle m\rangle}$ is trim. Moreover, if $\mathbf{w},\mathbf{w}'\in C_k^{\langle m\rangle}$ with $\mathbf{w}\leq_k\mathbf{w}'$, then we have

$$\mu_{\mathcal{C}_k^{\langle m\rangle}}(\mathbf{w},\mathbf{w}')=\begin{cases}1, & \textit{if } [\mathbf{w},\mathbf{w}'] \textit{ is nuclear and has two atoms},\\ -1, & \textit{if } \mathbf{w}\lessdot_k\mathbf{w}',\\ 0, & \textit{otherwise}.\end{cases}$$

PROOF. First we show that every element in $\mathcal{C}_k^{\langle m\rangle}$ has at most two upper covers. Let $\mathbf{w}\in C_k^{\langle m\rangle}$. We distinguish three cases:

(i) Let $\mathbf{w}=\left(\hat{0}^{l_0},a_i^{l_1},a_{i+1}^{l_2}\right)$ for $i\in\{1,2,\ldots,k-1\}$, where a_k is to be interpreted as $\hat{1}$. Since a_{i+1} is the unique upper cover of a_i, and $a_i\parallel_k b$ it follows that $\left(\hat{0}^{l_0-1},a_i^{l_1+1},a_{i+1}^{l_2}\right)$ and $\left(\hat{0}^{l_0},a_i^{l_1-1},a_{i+1}^{l_2+1}\right)$ are the only possible upper covers of $\mathbf{w}$.

(ii) Let $\mathbf{w}=\left(\hat{0}^{l_0},b^{l_1},\hat{1}^{l_2}\right)$. Again since $\hat{1}$ is the unique upper cover of b and since $b\parallel_k a_i$ for all $i\in\{1,2,\ldots,k-1\}$, it follows that $\left(\hat{0}^{l_0-1},b^{l_1+1},\hat{1}^{l_2}\right)$ and $\left(\hat{0}^{l_0},b^{l_1-1},\hat{1}^{l_2+1}\right)$ are the only possible upper covers of $\mathbf{w}$.

(iii) Let $\mathbf{w}=\left(\hat{0}^m\right)$. Since a_1 and b are the only atoms of $\mathcal{C}_k$ it follows that $\left(\hat{0}^{m-1},a_1\right)$ and $\left(\hat{0}^{m-1},b\right)$ are the only upper covers of $\mathbf{w}$.

Next we show that $\mathcal{C}_k^{\langle m\rangle}$ is trim. Since $k>1$ it follows that $\hat{0}\notin\mathcal{M}(\mathcal{C}_k)$ and $\hat{1}\notin\mathcal{J}(\mathcal{C}_k)$, and since $\mathcal{C}_k$ is extremal Corollary 2.4.7 implies that $\mathcal{C}_k^{\langle m\rangle}$ is extremal. Let $\mathbf{a}_{0,m}=\left(\hat{0}^m\right)$, and define $\mathbf{a}_{i,j}=\left(a_{i-1}^{m-j},a_i^j\right)$ for $i\in\{1,2,\ldots,k\}$ and $j\in\{1,2,\ldots,m\}$. (Again, a_0 and a_k are to be interpreted as $\hat{0}$ and $\hat{1}$, respectively.) Then it follows from the proof of Proposition 2.4.4 that $\mathbf{a}_{0,m}\lessdot_k\mathbf{a}_{1,1}\lessdot_k\cdots\lessdot_k\mathbf{a}_{k,m}$ is indeed a maximal chain in $\mathcal{C}_k^{\langle m\rangle}$. We show that $\mathbf{a}_{i,j}$ is left-modular for $i\in\{1,2,\ldots,k\}$ and $j\in\{1,2,\ldots,m\}$. (The element $\mathbf{a}_{0,m}$ is the least element of $\mathcal{C}_k^{\langle m\rangle}$ and thus trivially left-modular.) Fix $i\in\{1,2,\ldots,k\}$ and $j\in\{1,2,\ldots,m\}$. In view of Theorem 1.1.21 it suffices to consider elements $\mathbf{w},\mathbf{w}'\in C_k^{\langle m\rangle}$ with $\mathbf{w}\lessdot_k\mathbf{w}'$. We distinguish two cases:

(i) Let $\mathbf{w}'=\left(\hat{0}^{l_0},a_s^{l_1},a_{s+1}^{l_2}\right)$ for $s\in\{1,2,\ldots,k-1\}$ and $l_0<m$. Then we have essentially two choices for $\mathbf{w}$, namely $\mathbf{w}_1=\left(\hat{0}^{l_0-1},a_s^{l_1+1},a_{s+1}^{l_2}\right)$ or $\mathbf{w}_2=\left(\hat{0}^{l_0},a_s^{l_1-1},a_{s+1}^{l_2+1}\right)$. We have

$$\mathbf{a}_{i,j}\wedge_k\mathbf{w}'=\begin{cases}\left(\hat{0}^{l_0},a_s^{l_1},a_{s+1}^{l_2}\right), & \text{if } s<i,\\ \left(\hat{0}^{l_0},a_s^{m-l_0-\min\{j,l_2\}},a_{s+1}^{\min\{j,l_2\}}\right), & \text{if } s=i,\\ \left(\hat{0}^{l_0},a_s^{m-l_0-j},a_{s+1}^{j}\right), & \text{if } s>i.\end{cases}$$

It follows that $\mathbf{a}_{i,j}\wedge_k\mathbf{w}_1\neq\mathbf{a}_{i,j}\wedge_k\mathbf{w}'$ and $\mathbf{a}_{i,j}\wedge_k\mathbf{w}_2\neq\mathbf{a}_{i,j}\wedge_k\mathbf{w}'$ if and only if $s<i$ or $s=i$ and $l_2<j$. On the other hand, we have

$$\mathbf{a}_{i,j}\vee_k\mathbf{w}'=\begin{cases}\bigl(a_i^{m-j},a_{i+1}^{j}\bigr), & \text{if } s<i,\\ \bigl(a_s^{m-\max\{j,l_2\}},a_{s+1}^{\max\{j,l_2\}}\bigr), & \text{if } s=i,\\ \bigl(a_s^{m-l_2},a_{s+1}^{l_2}\bigr), & \text{if } s>i.\end{cases}$$

Here it follows that $\mathbf{a}_{i,j}\vee_k\mathbf{w}_1=\mathbf{a}_{i,j}\vee_k\mathbf{w}'$ and $\mathbf{a}_{i,j}\vee_k\mathbf{w}_2=\mathbf{a}_{i,j}\vee_k\mathbf{w}'$ if and only if $s<i$ or $s=i$ and $l_2<j$.

(ii) Let $\mathbf{w}'=\bigl(\hat{0}^{l_0},b^{l_1},\hat{1}^{l_2}\bigr)$ with $l_0,l_2<m$. Again, we have essentially two choices for $\mathbf{w}$, namely $\mathbf{w}_1=\bigl(\hat{0}^{l_0-1},b^{l_1+1},\hat{1}^{l_2}\bigr)$ or $\mathbf{w}_2=\bigl(\hat{0}^{l_0},b^{l_1-1},\hat{1}^{l_2+1}\bigr)$. We have

$$\mathbf{a}_{i,j}\wedge_k\mathbf{w}'=\begin{cases}\bigl(\hat{0}^{m-l_2},a_{i+1}^{l_2}\bigr), & \text{if } l_2\leq j,\\ \bigl(\hat{0}^{m-l_2},a_i^{l_2-j},a_{i+1}^{j}\bigr), & \text{if } l_2>j.\end{cases}$$

It follows that $\mathbf{a}_{i,j}\wedge_k\mathbf{w}_1=\mathbf{a}_{i,j}\wedge_k\mathbf{w}'$ and $\mathbf{a}_{i,j}\wedge_k\mathbf{w}_2\neq\mathbf{a}_{i,j}\wedge_k\mathbf{w}'$. On the other hand, we have

$$\mathbf{a}_{i,j}\vee_k\mathbf{w}'=\bigl(a_{k-1}^{l_0},\hat{1}^{m-l_0}\bigr).$$

Here it follows that $\mathbf{a}_{i,j}\vee_k\mathbf{w}_1\neq\mathbf{a}_{i,j}\vee_k\mathbf{w}'$ and $\mathbf{a}_{i,j}\vee_k\mathbf{w}_2=\mathbf{a}_{i,j}\vee_k\mathbf{w}'$.

Thus $\mathbf{a}_{i,j}$ is left-modular, and so is $\mathcal{C}_k^{\langle m\rangle}$, which implies with its extremality that it is trim.

Finally we compute the values of the Möbius function of $\mathcal{C}_k^{\langle m\rangle}$. Let $\mathbf{w},\mathbf{w}'\in C_k^{\langle m\rangle}$ with $\mathbf{w}\leq_k\mathbf{w}'$. First suppose that $[\mathbf{w},\mathbf{w}']$ is nuclear. Then Theorem 1.1.24 implies that $\mu_{\mathcal{C}_k^{\langle m\rangle}}(\mathbf{w},\mathbf{w}')=(-1)^s$, where s is the number of atoms of $[\mathbf{w},\mathbf{w}']$. In view of the first part of this proof it follows that $s\leq 2$. If $s=1$, then necessarily $\mathbf{w}\lessdot_k\mathbf{w}$, because we assumed $[\mathbf{w},\mathbf{w}']$ to be nuclear. If $[\mathbf{w},\mathbf{w}']$ is not nuclear, then Theorem 1.1.24 implies that $\mu_{\mathcal{C}_k^{\langle m\rangle}}(\mathbf{w},\mathbf{w}')=0$. Hence the proof is complete. $\square$

CHAPTER 3

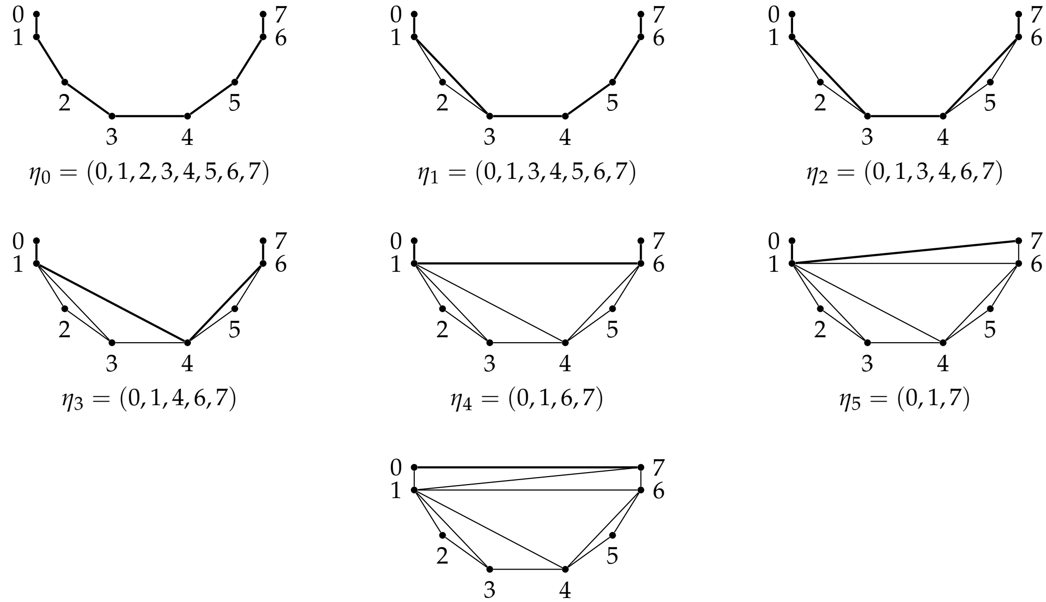
The Cambrian Lattices

3.1. Introduction

Björner and Wachs observed in [**26**, Theorem 9.6] that the Tamari lattice $\mathcal{T}_n$ is isomorphic to the lattice of 312-avoiding permutations of $\{1,2,\ldots,n\}$ under (right) weak order. This observation led Reading in [**93**] to the construction of a generalization of the Tamari lattices different from the m-Tamari lattices considered in the previous chapter. In type A the key observation for this generalization is that each permutation of $\{1,2,\ldots,n\}$ induces a triangulation of a regular $(n+2)$-gon as follows. We denote the $(n+2)$-gon by Q_n, and we label its nodes by $0,1,\ldots,n+1$. We draw the nodes 0 and $n+1$ on a horizontal line, and for $i\in\{1,2,\ldots,n\}$ we place the node i in such a way strictly between the nodes $i-1$ and $i+1$ that we obtain a convex polygon in which no three nodes are collinear. Then, we define $\eta_0=(0,1,\ldots,n+1)$, and we interpret it as drawing a path from 0 to $n+1$ by connecting consecutive entries in η_0. Let $\pi\in A_{n-1}$ be a permutation, where $\pi=\pi_1\pi_2\cdots\pi_n$ is its one-line notation, i.e. $\pi_i=\pi(i)$ for $i\in\{1,2,\ldots,n\}$. We construct η_i from η_{i-1} by removing π_i, and again we interpret η_i as drawing a path from 0 to $n+1$ through the nodes contained in η_i. Since $\pi\in A_n$, it follows that $\eta_n=(0,n+1)$. If we now superimpose the paths $\eta_0,\eta_1,\ldots,\eta_n$ on Q_n, then we obtain a triangulation of Q_n. See Figure 25 for an illustration. We denote the set of all triangulations of Q_n by $\Delta(Q_n)$, and hence there is a well-defined map from A_{n-1} to $\Delta(Q_n)$ that we also denote by η.

In order to recover the Tamari lattice as a poset on triangulations, we need to define a suitable partial order. It is obvious that each triangulation $D\in\Delta(Q_n)$ is completely characterized by its *diagonals*, namely lines connecting two nodes i and j with $i\not\equiv j\pm 1 \pmod{n+2}$. If we remove a diagonal d from D, then we obtain a quadrilateral, and if we put the opposite diagonal d' back in, then we obtain a different triangulation $D'\in\Delta(Q_n)$. If the slope of d' (with respect to the embedding of Q_n in the plane described in the previous paragraph) is larger than the slope of d, then we interpret such a move as going up one step. Hence we can define a partial order on $\Delta(Q_n)$, denoted by $\leq_{\text{flip}}$, where the cover relations are diagonal flips. See Figure 26 for an illustration.

However, we notice that there are $n!$ elements in A_{n-1}, but only $\mathrm{Cat}(n)$ elements in $\Delta(Q_n)$, see [**113**, Exercise 6.19(a)]. Hence the map η is certainly not injective, but it is easy to see that η is surjective. But more can be said. Recall that a *fiber* of a map $f:M\to N$ is the preimage $f^{-1}(n)=\{m\in M\mid f(m)=n\}$ for some $n\in N$, and recall that a *lattice congruence* of a lattice

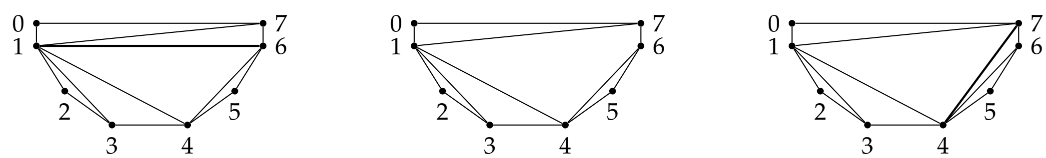


$\eta_0 = (0,1,2,3,4,5,6,7)$ $\eta_1 = (0,1,3,4,5,6,7)$ $\eta_2 = (0,1,3,4,6,7)$

$\eta_3 = (0,1,4,6,7)$ $\eta_4 = (0,1,6,7)$ $\eta_5 = (0,1,7)$

$\eta_6 = (0,7)$

FIGURE 25. A triangulation of Q_6 derived from the permutation 253461.



A triangulation $D \in \Delta(Q_6)$. Removing the diagonal $(1,6)$ from D. Putting the opposite diagonal $(4,7)$ back in.

FIGURE 26. Illustration of a diagonal flip.

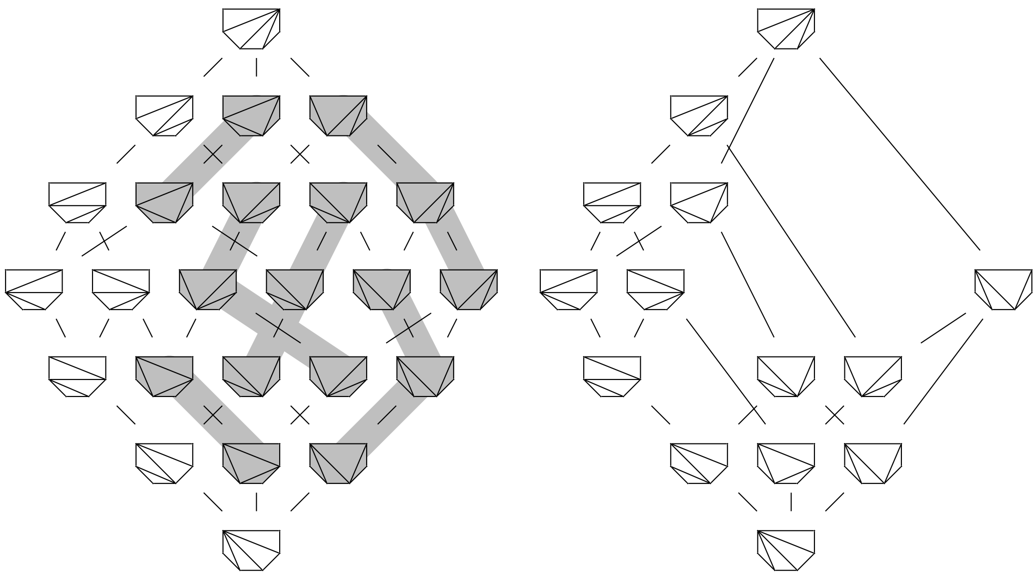
$\mathcal{P} = (P, \leq)$ is an equivalence relation θ on P respecting joins and meets, i.e. if $p_1 \ \theta \ p_2$ and $q_1 \ \theta \ q_2$ for $p_1, p_2, q_1, q_2 \in P$, then $(p_1 \wedge q_1) \ \theta \ (p_2 \wedge q_2)$ and $(p_1 \vee q_1) \ \theta \ (p_2 \vee q_2)$.

THEOREM 3.1.1 ([93, Theorem 5.1])

The fibers of η are the congruence classes of a lattice congruence θ on the weak order on A_{n-1}. In particular, η is a surjective lattice homomorphism. Moreover, an element $\pi \in A_{n-1}$ is the least element in a congruence class of θ if and only if it is 312-avoiding.

EXAMPLE 3.1.2

Figure 27(a) shows the weak order lattice of A_3, where the nodes are labeled by the triangulations induced via the map η. The non-singleton congruence classes with respect to the congruence relation mentioned in Theorem 3.1.1 are highlighted. Figure 27(b) shows the corresponding sublattice of 312-avoiding permutations and this lattice is indeed isomorphic to $\mathcal{T}_4$.



(a) The weak order lattice on A_3 realized as a poset on triangulations of a convex 6-gon.

(b) The sublattice $\big(\Delta(Q_4),\leq_{\mathrm{flip}}\big)$.

FIGURE 27. Two posets on triangulations of a convex 6-gon.

The next step in READING's generalization of the Tamari lattices is the observation that different embeddings of Q_n in the plane yield different lattice congruences of the weak order on A_{n-1}. More precisely, let $f:\{1,2,\ldots,n\}\to\{-1,1\}$ be some map, and embed Q_n in the plane as follows. Again we draw the nodes 0 and $n+1$ on a horizontal line, and for $i\in\{1,2,\ldots,n\}$ we draw the node i strictly between the nodes $i-1$ and $i+1$. Now, however, we draw the node i below the line connecting 0 and $n+1$ if and only if $f(i)=-1$, and we draw it above otherwise, and we denote the resulting polygon by $Q_n^{(f)}$. Then, we derive a triangulation of $Q_n^{(f)}$ from a permutation $\pi\in A_{n-1}$ with one-line notation $\pi_1\pi_2\cdots\pi_n$ similarly to before: let $O=\big\{i\in\{1,2,\ldots,n\}\mid f(i)=1\big\}=\{o_1,o_2,\ldots,o_k\}$, and let $U=\big\{i\in\{1,2,\ldots,n\}\mid f(i)=-1\big\}=\{u_1,u_2,\ldots,u_{n-k}\}$. Let $\eta_0^{(f)}=(0,u_1,u_2,\ldots,u_{n-k},n+1)$. We construct $\eta_i^{(f)}$ from $\eta_{i-1}^{(f)}$ by adding the index i if and only if $f(i)=1$ and by removing it otherwise. It is clear that $\eta_n^{(f)}=(0,o_1,o_2,\ldots,o_k,n+1)$, and we denote the map from A_{n-1} to $\Delta\big(Q_n^{(f)}\big)$ by $\eta^{(f)}$. See Figure 28 for an illustration. The partial order remains the same as before, namely going up one step by flipping a diagonal and increasing the slope. READING proved the following result.

THEOREM 3.1.3 ([93, Theorem 5.1])

The fibers of $\eta^{(f)}$ are the congruence classes of a lattice congruence $\theta^{(f)}$ on the weak order on A_n. In particular, $\eta^{(f)}$ is a surjective lattice homomorphism.

Figure 29 shows the poset $\big(\Delta\big(Q_6^{(f)}\big),\leq_{\mathrm{flip}}\big)$, where $f(1)=f(2)=-1$ and $f(3)=f(4)=1$. If we compare Figures 27(b) and 29, then we observe that varying the map f yields different

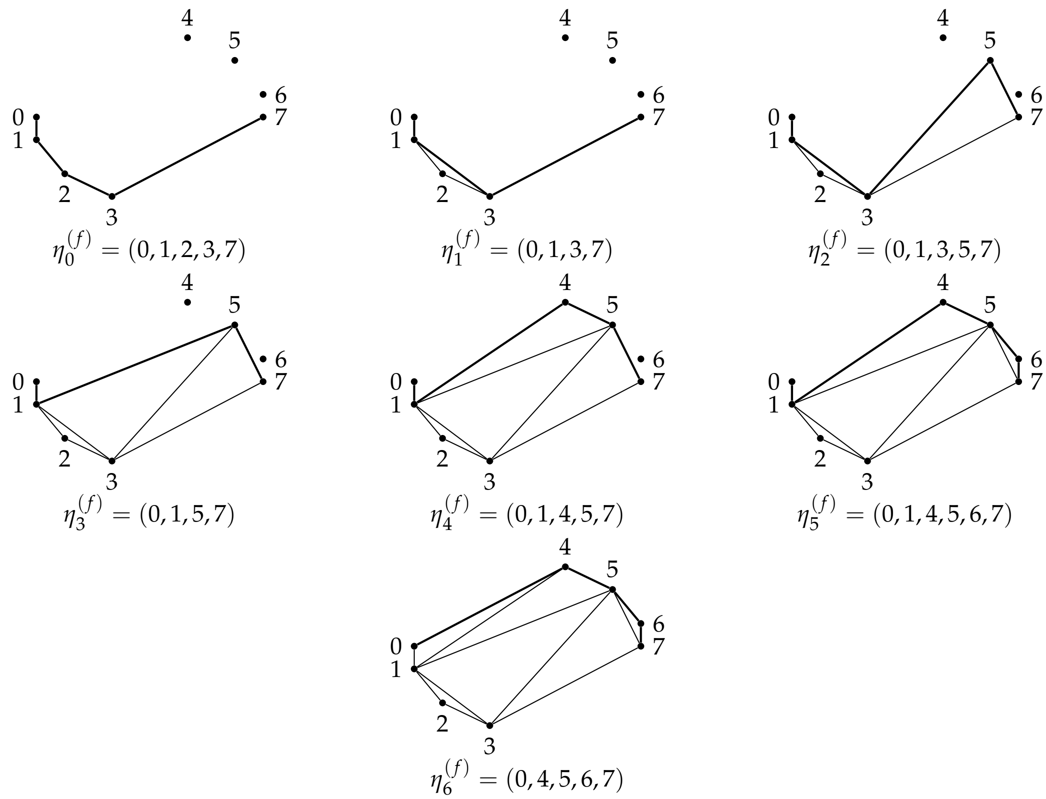


FIGURE 28. A triangulation of $Q_6^{(f)}$, where $f(1) = f(2) = f(3) = -1$ and $f(4) = f(5) = f(6) = 1$, derived from the permutation 253461.

sublattices of the weak order, in the sense that they are in general nonisomorphic. However, these sublattices are "combinatorially isomorphic" in the following sense: their Hasse diagrams can be seen as the 1-*skeleton* of an $(n-1)$-dimensional polytope, i.e. their vertices correspond to the 0-dimensional faces of some polytope, and their cover relations correspond to the 1-dimensional faces of the same polytope. It turns out that this polytope does not depend on the map f, see [**93**, Theorem 1.3]. In particular, this polytope is the well-known *associahedron*. The fact that the Hasse diagram of the Tamari lattice is the 1-skeleton of the associahedron was already observed by TAMARI. Later, STASHEFF independently (re)discovered the associahedra in the context of H-spaces and the study of associativity up to homotopy, see [**114**,**115**]. Since then, analogously to the Tamari lattices, the associahedra have been attractive objects of research, and we refer again to [**86**] for a recent overview on the appearances of the associahedra in many different fields of mathematics.

The maps f from before do not only induce embeddings of a convex $(n+2)$-gon in the plane, they are also in correspondence with the Coxeter elements of A_{n-1}. Recall that the Coxeter diagram of A_{n-1} is a path with $n-2$ edges. For $i \in \{1,2,\ldots,n-2\}$, we orient the edge between s_i and s_{i+1} from left to right if and only if $f(i+1) = -1$, and from right to left otherwise. Thus each f yields a (not necessarily different) orientation of the Coxeter diagram of A_{n-1}, and these orientations are in bijection with the Coxeter elements of A_{n-1}, see [**105**]. See Figure 30 for an illustration.

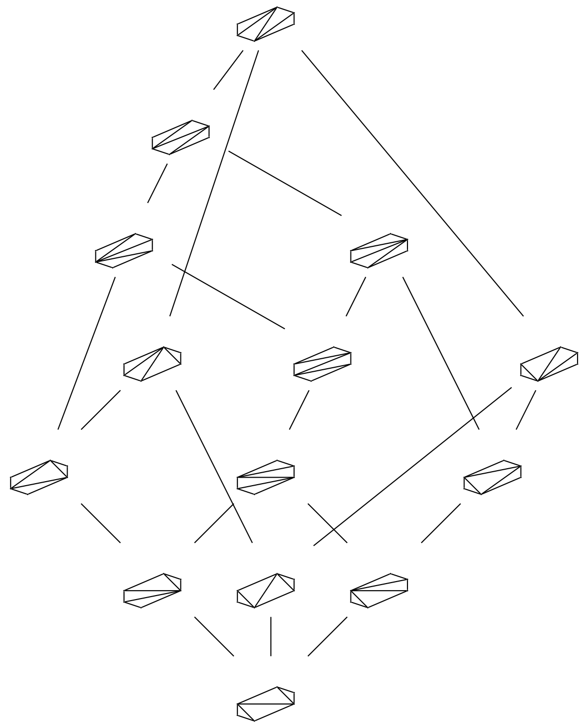
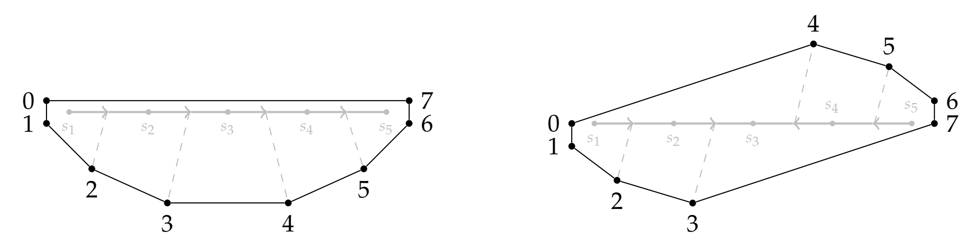


FIGURE 29. The poset $\big(\Delta(Q_4^{(f)}), \leq_{\text{flip}}\big)$, where $f(1) = f(2) = 1$ and $f(3) = f(4) = -1$.



(a) $f(1) = f(2) = f(3) = f(4) = f(5) = f(6) = -1$.

(b) $f(1) = f(2) = f(3) = -1, f(4) = f(5) = f(6) = 1$.

FIGURE 30. Two embeddings of a convex 8-gon and the corresponding orientation of Γ_{A_5}.

Type-B analogues of the associahedra have been proposed in [29, 107, 116]. More generally, in [54] generalized associahedra have been defined for all Weyl groups in the context of cluster algebras. A type-B Tamari lattice was introduced in [122] as a poset on centrally symmetric triangulations, where the partial order is given by diagonal flips. In [93], READING showed that, analogously to the case of ordinary triangulations, different embeddings of the underlying polygon yield different sublattices of the weak order of the Coxeter group of type B. These ideas led READING to the definition of so-called γ-sortable elements for an arbitrary Coxeter group W and some Coxeter element $\gamma \in W$, see [95, 97] This construction in turn defined generalized associahedra simultaneously for every finite Coxeter group. Moreover, READING's γ-sortable elements constitute a sub-semilattice of the weak order semilattice of W. Thus they provide a generalization of the Tamari lattices to all Coxeter groups. These semilattices, the so-called γ-Cambrian semilattices, are the object of interest in this chapter.

Here we investigate some topological and structural properties of the γ-Cambrian semilattices. In particular, we prove that these semilattices are EL-shellable, and we compute their Möbius function, see Theorem 3.4.1. Moreover, we prove that the γ-Cambrian semilattices are trim, and that they are bounded-homomorphic images of some free lattice, see Theorems 3.4.14 and 3.5.1. All of these results generalize known properties of the Tamari lattices, and they are obtained in a uniform way, i.e. they are obtained simultaneously for all Coxeter groups and all Coxeter elements.

3.2. Definition and Examples

Let us now formally define the γ-Cambrian semilattices associated with a Coxeter system (W,S) of rank n and a Coxeter element $\gamma \in W$. Let $S = \{s_1, s_2, \ldots, s_n\}$ and recall that $\gamma = s_{\sigma_1}s_{\sigma_2}\cdots s_{\sigma_n}$ for some permutation σ of $\{1,2,\ldots,n\}$. Without loss of generality, we can assume σ to be the identity and hence $\gamma = s_1s_2\cdots s_n$. Consider the half-infinite word

$$\gamma^\infty = s_1s_2\cdots s_n|s_1s_2\cdots s_n|s_1\cdots, \tag{3.1}$$

where the vertical bars serve only for indicating the repetitions of γ and do not have any influence on the structure of the word. Since S generates W every reduced decomposition of some $w \in W$ can be written as a subword of γ^∞, and among all reduced decompositions of w there is a unique reduced decomposition that is lexicographically first as a subword of γ^∞. We call this reduced decomposition the *γ-sorting word of w*, and we denote it by $\gamma(w)$. We can write

$$\gamma(w) = s_1^{\delta_{1,1}}s_2^{\delta_{1,2}}\cdots s_n^{\delta_{1,n}}|s_1^{\delta_{2,1}}s_2^{\delta_{2,2}}\cdots s_n^{\delta_{2,n}}|\cdots|s_1^{\delta_{l,1}}s_2^{\delta_{l,2}}\cdots s_n^{\delta_{l,n}}, \tag{3.2}$$

for some $l \in \mathbb{N}$ with $\delta_{i,j} \in \{0,1\}$ for $i \in \{1,2,\ldots,n\}$ and $j \in \{1,2,\ldots,l\}$. Now for $i \in \{1,2,\ldots,l\}$, we call the set

$$b_i = \{s_j \mid j \in \{1,2,\ldots,n\} \text{ and } \delta_{i,j} = 1\} \tag{3.3}$$

the *i-th block of w*[1].

DEFINITION 3.2.1

Let (W,S) be a Coxeter system of rank n, where $S = \{s_1, s_2, \ldots, s_n\}$, and let $\gamma = s_1s_2\cdots s_n \in W$ be a Coxeter element of W. We call an element $w \in W$ *γ-sortable* if the sequence of blocks of w is weakly decreasing with respect to inclusion, i.e. $b_1 \supseteq b_2 \supseteq \cdots \supseteq b_l$. We write C_γ for the set of γ-sortable elements of W.

REMARK 3.2.2

It follows from [**94**, Theorem 4.12] that a permutation $\pi \in A_{n-1}$ is γ-sortable for $\gamma = s_1s_2\cdots s_{n-1}$ with $s_i = (i\ i{+}1)$ for $i \in \{1,2,\ldots,n-1\}$ if and only if it is 312-avoiding.

[1] Note that we speak of a "block of w", when we actually mean a "block of the γ-sorting word of w". Since we will always consider the γ-sorting words of the elements of w, we use this abbreviated notion.

Example 3.2.3

Let $\pi = 21543 \in A_4$, and consider the Coxeter element $\gamma = s_1s_2s_3s_4 = 23451$. There are eight reduced decompositions of π, namely

$$s_1s_4s_3s_4,\quad s_4s_1s_3s_4,\quad s_4s_3s_1s_4,\quad s_4s_3s_4s_1,\quad s_1s_3s_4s_3,\quad s_3s_1s_4s_3,\quad s_3s_4s_1s_3,\quad s_3s_4s_3s_1,$$

and we can quickly verify that $\gamma(\pi) = s_1s_3s_4s_3$. Thus we can write

$$\gamma(\pi) = s_1^1s_2^0s_3^1s_4^1|s_1^0s_2^0s_3^1s_4^0,$$

and the blocks of π are $b_1 = \{s_1, s_3, s_4\}$ and $b_2 = \{s_3\}$. Since $b_1 \supseteq b_2$ it follows that π is γ-sortable. In fact, we can also check that π is 312-avoiding.

On the other hand let $\pi' = 24153 \in A_4$. There are five reduced decompositions of π', namely

$$s_1s_3s_4s_2,\quad s_1s_3s_2s_4,\quad s_3s_1s_2s_4,\quad s_3s_1s_4s_2,\quad s_3s_4s_1s_2,$$

and we can quickly verify that $\gamma(\pi') = s_1s_3s_4s_2$. We have

$$\gamma(\pi') = s_1^1s_2^0s_3^1s_4^1|s_1^0s_2^1s_3^0s_4^0,$$

and the blocks of π' are $b_1 = \{s_1, s_3, s_4\}$ and $b_2 = \{s_2\}$. Now we see that $b_1 \not\supseteq b_2$, which implies that π' is not γ-sortable. Moreover, we can check that π' is not 312-avoiding, since $\pi_2 = 4, \pi_3 = 1$ and $\pi_5 = 3$.

The γ-sortable elements of W possess an intrinsic recursive structure. In order to explain this structure, we recall that some of the subgroups of W are Coxeter groups again (not necessarily of the same type, though). More precisely, let (W, S) be a Coxeter system, and let $J \subseteq S$. Then, J generates a *standard parabolic subgroup* W_J of W, and it is immediate that (W_J, J) is a Coxeter system itself. Moreover, the Coxeter diagram $\Gamma_{(W_J,J)}$ is the subgraph of $\Gamma_{(W,S)}$ induced by the vertices in J. Any subgroup of W that is conjugate to a standard parabolic subgroup is called a *parabolic subgroup of W*. In the context of γ-sortable elements, we will frequently need the case, where we reduce the rank of W only by one, hence where $J = S \setminus \{s\}$ for some $s \in S$. In this case, we will use the abbreviation $\langle s\rangle = S \setminus \{s\}$.

If we define $W^J = \{w \in W \mid \ell(ws) > \ell(w) \text{ for all } s \in J\}$, then we have the following decomposition of the elements in W.

Proposition 3.2.4 ([23, Proposition 2.4.4])

Let (W, S) be a Coxeter system, and let $J \subseteq S$. For every $w \in W$ there exists a unique reduced decomposition $w = w^J \cdot w_J$ such that $w^J \in W^J$ and $w_J \in W_J$. Moreover, we have $\ell_S(w) = \ell_S(w^J) + \ell_S(w_J)$.

Hence we can interpret w_J as “the part of w which lies in W_J”, and it follows that $\mathrm{inv}(w_J) = \mathrm{inv}(w) \cap W_J$. In particular, $u \leq_S v$ implies $u_J \leq_J v_J$. In fact, joins and meets in $\mathcal{W}$ are compatible with the restriction to parabolic subgroups.

Proposition 3.2.5 ([65, Lemmas 4.2(iii) and 4.5])

Let (W, S) be a Coxeter system, let $J \subseteq S$, and let $A \subseteq W$. Define $A_J = \{w_J \mid w \in A\}$. If A is nonempty, then $\bigwedge A_J = \big(\bigwedge A\big)_J$, and if A has an upper bound, then $\bigvee A_J = \big(\bigvee A\big)_J$.

But let us now return to the recursive structure of the γ-sortable elements of W. We say that $s \in S$ is an *initial letter of γ* if there exists a reduced decomposition of γ that starts with s.

PROPOSITION 3.2.6 ([97, Proposition 2.29])

Let (W,S) be a Coxeter system, let $\gamma \in W$ be a Coxeter element, and let $s \in S$ be an initial letter of γ. Then, $w \in W$ is γ-sortable if and only if it satisfies one of the following two properties:

(i) if $s \leq_S w$, then sw is $s\gamma s$-sortable in W; or
(ii) if $s \not\leq_S w$, then w is $s\gamma$-sortable in $W_{\langle s\rangle}$.

REMARK 3.2.7

The definition of γ-sortable elements does not depend on the choice of reduced decomposition of γ. Since each reduced decompositions of γ can be transformed into any other reduced decomposition of γ by commutations of letters, it is clear that such commutations leave the blocks of any $w \in W$ fixed.

Proposition 3.2.6 implies that we can apply two parallel inductions, when working with γ-sortable elements. On the one hand we can apply induction on the length of w, and on the other hand we can apply induction on the rank of W, and these inductions do not interfere with each other. Now we are set to define the γ-Cambrian semilattices.

DEFINITION 3.2.8

Let W be a Coxeter group, and let $\gamma \in W$ be a Coxeter element. The poset $\mathcal{C}_\gamma = (C_\gamma, \leq_\gamma)$ is called the *γ-Cambrian semilattice of W*, where $\leq_\gamma$ denotes the restriction of $\leq_S$ to C_γ.

In the remainder of this chapter we will frequently switch between the weak order and the Cambrian order, i.e. the restriction of the weak order to C_γ. In order to properly distinguish which situation we currently consider, we add a subscript "S" to poset-theoretic notions, when we consider the weak order, and we add a subscript "γ" to poset-theoretic notions, when we consider the Cambrian order. For instance, a closed interval in $\mathcal{W}$ will be denoted by $[u,v]_S$, while a closed interval in $\mathcal{C}_\gamma$ will be denoted by $[u,v]_\gamma$. Analogously, we distinguish open intervals, joins, meets, cover relations, and so on. Figures 31 and 32 show two γ-Cambrian lattices, the first one associated with the Coxeter group A_3, and the second one associated with the Coxeter group B_3.

3.3. Basic Properties

Now we describe some basic properties of the γ-Cambrian semilattices, and we start with the observation that $\mathcal{C}_\gamma$ is in fact a meet-subsemilattice of $\mathcal{W}$.

THEOREM 3.3.1 ([97, Theorem 7.1])

Let W be a Coxeter group, let $\gamma \in W$ be a Coxeter element, and let $A \subseteq C_\gamma$. If A is nonempty, then $\bigwedge_S A$ is γ-sortable. If A has an upper bound, then $\bigvee_S A$ is γ-sortable.

Analogously to the maps η_f described in the introduction of this chapter, there exists a lattice homomorphism from the weak order semilattice to the Cambrian lattice. This map is defined according to the recursive structure of the γ-sortable elements described in Proposition 3.2.6, namely by

$$\pi_\downarrow^\gamma : W \to C_\gamma, \quad w \mapsto \begin{cases} s\pi_\downarrow^{s\gamma s}(sw), & \text{if } s \leq_S w, \\ \pi_\downarrow^{s\gamma}(w_{\langle s\rangle}), & \text{if } s \not\leq_S w. \end{cases}, \tag{3.4}$$

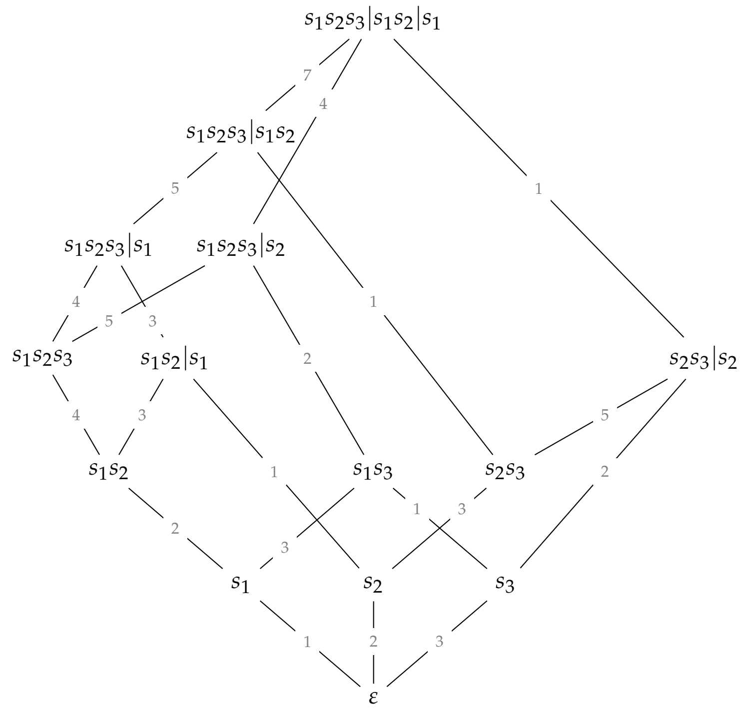


FIGURE 31. An A_3-Cambrian lattice with the labeling defined in (3.7). This lattice is in fact isomorphic to $\mathcal{T}_4$.

for some initial letter s of γ, and where we set $\pi_{\downarrow}^{\gamma}(\varepsilon) = \varepsilon$ for all Coxeter elements γ in all Coxeter groups. The most important properties of this map are the following.

THEOREM 3.3.2 ([97, Theorem 6.1])

The map $\pi_{\downarrow}^{\gamma}$ is order-preserving for any Coxeter group W and any Coxeter element $\gamma \in W$.

THEOREM 3.3.3 ([97, Theorem 7.3])

Let W be a Coxeter group, let $\gamma \in W$ be a Coxeter element, and let $A \subseteq W$. If A is nonempty, then $\bigwedge_{\gamma}\pi_{\downarrow}^{\gamma}(A) = \pi_{\downarrow}^{\gamma}(\bigwedge_S A)$. Moreover, if A has an upper bound, then $\bigvee_{\gamma}\pi_{\downarrow}^{\gamma}(A) = \pi_{\downarrow}^{\gamma}(\bigvee_S A)$.

Now we recall some structural properties of $\mathcal{C}_{\gamma}$.

THEOREM 3.3.4 ([97, Theorem 8.1])

Let W be a Coxeter group. Then, every $w \in W$ has a canonical join-representation Z_w in $\mathcal{W}$. Furthermore, $\mathrm{cov}(w)$ *is the disjoint union of* $\mathrm{cov}(j)$ *for $j \in Z_w$.*

PROPOSITION 3.3.5 ([97, Proposition 8.2])

Let W be a Coxeter group, and let $\gamma \in W$ be a Coxeter element. If $w \in C_{\gamma}$, then every element of its canonical join-representation is γ-sortable.

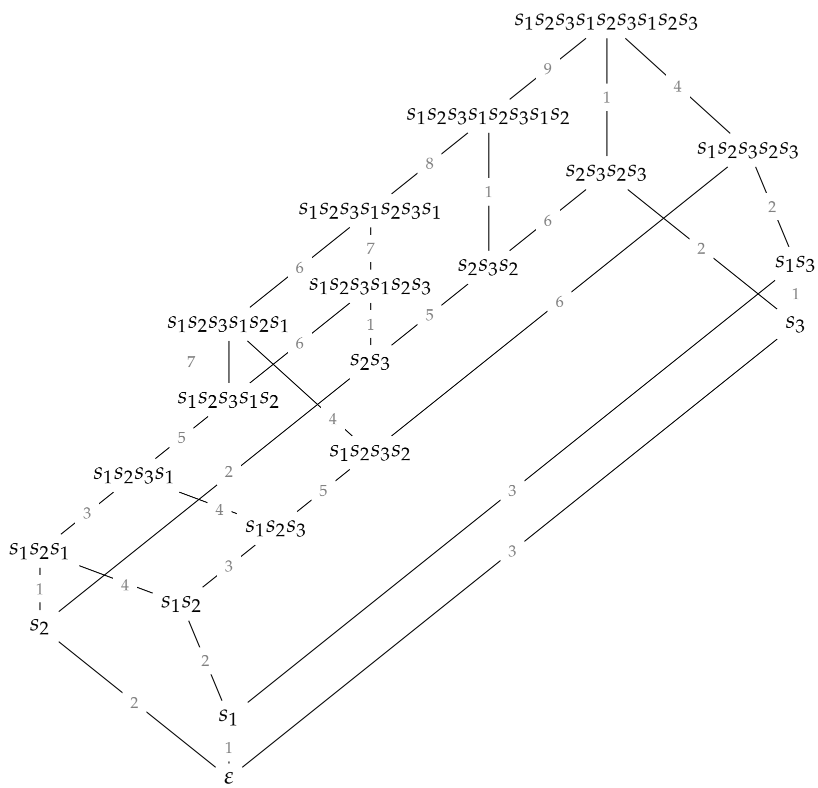


FIGURE 32. A B_3-Cambrian lattice with the labeling defined in (3.7).

As a consequence we obtain that $\mathcal{C}_\gamma$ is semidistributive, which is already implicit in [**97**, Section 8].

PROPOSITION 3.3.6

Let W be a Coxeter group, and let $\gamma \in W$ be a Coxeter element. Then, every closed interval of $\mathcal{C}_\gamma$ is semidistributive.

PROOF. Let $w \in C_\gamma$. Lemma 1.1.29 and Theorem 3.3.4 imply that the interval $[\varepsilon, w]_S$ is join-semidistributive. Let $x, y, z \in C_\gamma$ with $x, y, z \leq_S w$, and suppose that $x \vee_\gamma y = x \vee_\gamma z$. Theorem 3.3.1 implies then, that $x \vee_S y = x \vee_S z$, and it follows from the join-semidistributivity of $\mathcal{W}$ and Theorem 3.3.3 that

$$\begin{aligned} x \vee_\gamma y = \pi_\downarrow^\gamma(x) \vee_\gamma \pi_\downarrow^\gamma(y) = \pi_\downarrow^\gamma(x \vee_S y) = \pi_\downarrow^\gamma\big(x \vee_S (y \wedge_S z)\big) \\ = \pi_\downarrow^\gamma(x) \vee_\gamma \pi_\downarrow^\gamma(y \vee_S z) = \pi_\downarrow^\gamma(x) \vee_\gamma \big(\pi_\downarrow^\gamma(y) \wedge_\gamma \pi_\downarrow^\gamma(z)\big) = x \vee_\gamma (y \wedge_\gamma z). \end{aligned}$$

Thus the interval $[\varepsilon, w]_\gamma$ is join-semidistributive. Proposition 1.2.16 implies analogously that $[\varepsilon, w]_\gamma$ is meet-semidistributive. Thus $[\varepsilon, w]_\gamma$ is a semidistributive lattice. In view of the characterization of semidistributive lattices in Theorem 1.1.25, it follows that intervals of semidistributive lattices are semidistributive again and the proof is complete. $\square$

The semilattice $\mathcal{C}_\gamma$ is in general infinite, but Proposition 1.2.18 and Theorem 3.3.1 imply that the closed intervals of $\mathcal{C}_\gamma$ are finite lattices. Hence in what follows we focus on intervals of the form $[u,v]_\gamma$ for $u,v \in C_\gamma$ with $u \leq_\gamma v$. This study usually breaks down into three cases, namely (i) $s \leq_\gamma u$, (ii) $s \not\leq_\gamma u, s \leq_\gamma v$, and (iii) $s \not\leq_\gamma v$ for some initial letter s of γ. In view of Proposition 3.2.6, cases (i) and (iii) can be treated nicely by induction on length and rank. However, case (ii) requires some more preparation. We provide a simple lemma that helps reducing case (ii) to case (i).

LEMMA 3.3.7

Let (W,S) be a Coxeter system of rank n, where $S = \{s_1, s_2, \ldots, s_n\}$. Let $\gamma = s_1 s_2 \cdots s_n$, and let $u,v \in C_\gamma$ with $u \leq_\gamma v$. If $s_1 \not\leq_\gamma u$ and $s_1 \leq_\gamma v$, then the join $s_1 \vee_\gamma u$ covers u in $\mathcal{C}_\gamma$.

PROOF. First of all since $s_1 \leq_\gamma v$ and $u \leq_\gamma v$ it follows from Theorem 3.3.1 that $z = s_1 \vee_\gamma u$ exists. Moreover, for $w, w' \in C_\gamma$ with $w \leq_\gamma w'$ it follows from Theorem 3.3.2 that

$$w \leq_S w' \quad \text{implies} \quad \pi_\downarrow^\gamma(w) \leq_\gamma \pi_\downarrow^\gamma(w') \quad \text{implies} \quad w \leq_\gamma w'. \tag{3.5}$$

By assumption, we have $s_1 \not\leq_\gamma u$, and it follows then by contraposition from (3.5) that $s_1 \not\leq_S u$. Hence (3.4) implies $u = \pi_\downarrow^\gamma(u) = \pi_\downarrow^{s_1\gamma}(u_{\langle s_1\rangle}) \in W_{\langle s_1\rangle}$, and Proposition 3.2.6 implies $u = u_{\langle s_1\rangle} \in W_{\langle s_1\rangle}$.

Since $u <_\gamma z$ there exists some $u' \in C_\gamma$ with $u \leq_\gamma u' \lessdot_\gamma z$. If $s_1 \leq_\gamma u'$, then u' is an upper bound for both s_1 and u, which contradicts $u' \leq_\gamma z$. Thus we have $s_1 \not\leq_\gamma u'$, which with (3.5) implies $s_1 \not\leq_S u'$ again. Again Proposition 3.2.6 implies $u' \in W_{\langle s_1\rangle}$. Since $\mathcal{C}_\gamma$ is a sub-semilattice of $\mathcal{W}$ the relation $u' \lessdot_\gamma z$ implies $u' <_S z$, and we obtain $u'_{\langle s_1\rangle} \leq_S z_{\langle s_1\rangle} = (s_1 \vee_\gamma u)_{\langle s_1\rangle}$. In view of Proposition 3.2.5 this implies $u'_{\langle s_1\rangle} \leq_S (s_1)_{\langle s_1\rangle} \vee_\gamma u_{\langle s_1\rangle}$. However, since $(s_1)_{\langle s_1\rangle} = \varepsilon$ and $u_{\langle s_1\rangle} = u$ we conclude $u' \leq_S u$. With (3.5) follows $u = u'$ and thus the result. □

Let us record two more technical statements that will simplify later proofs.

LEMMA 3.3.8

Let $\gamma = s_1 s_2 \cdots s_n$, and let $u,v \in C_\gamma$ with $u \leq_\gamma$. If $s_1 \leq_\gamma u$, then the interval $[u,v]_\gamma$ is isomorphic to the interval $[s_1 u, s_1 v]_{s_1\gamma s_1}$ in $\mathcal{C}_{s_1\gamma s_1}$, and if $s_1 \not\leq_\gamma v$, then the interval $[u,v]_\gamma$ is isomorphic to the interval $\left[u_{\langle s_1\rangle}, v_{\langle s_1\rangle}\right]_{s_1\gamma}$ in $\mathcal{C}_{s_1\gamma}$.

PROOF. The statement for $s_1 \leq_\gamma u$ follows from Proposition 3.2.6 and Proposition 1.2.20. The statement for $s_1 \not\leq_\gamma v$ follows from Proposition 3.2.6 and from the observation that the weak order restricted to $W_{\langle s_1\rangle}$ yields a sub-semilattice of $\mathcal{W}$ (which itself is a consequence of Propositions 3.2.4 and 3.2.5). □

LEMMA 3.3.9

Let $\gamma = s_1 s_2 \cdots s_n \in W$, and let $u,v \in C_\gamma$ with $u \leq_\gamma v$. If $s_1 \not\leq_\gamma u$ and $s_1 \leq_\gamma v$, then the following are equivalent.

(i) The interval $[u,v]_\gamma$ is nuclear.
(ii) There exists an element $v' \in [u,v]_\gamma$ with $s_1 \not\leq_\gamma v' \lessdot_\gamma v$, and the interval $[u,v']_\gamma$ is nuclear.

PROOF. Let $A = \{x \in C_\gamma \mid u \lessdot_\gamma x \leq_\gamma v\}$ be the set of atoms in $[u,v]_\gamma$. Since $s_1 \leq_\gamma v$ and $u \leq_\gamma v$ it follows from Lemma 3.3.7 that the join $z = s_1 \vee_\gamma u$ covers u, and we set $A_z = A \setminus \{z\}$. The uniqueness of the existing joins in $\mathcal{C}_\gamma$ implies that $s_1 \not\leq_\gamma x$ for all $x \in A_z$. (Otherwise, suppose that there is some $x \in A_z$ with $s_1 \leq_\gamma x$. Then, x is an upper bound for both s_1 and

u, and hence $z \leq_\gamma x$. Since both z and x cover u, it follows that $x = z$, contradicting $z \notin A_z$.) Moreover, Proposition 3.2.6 implies $A_z \subseteq W_{\langle s_1 \rangle}$.

$(i) \Rightarrow (ii)$ Suppose that $[u,v]_\gamma$ is nuclear and let $v' = \bigvee_\gamma A_z$. The existence of v' is ensured by Theorem 3.3.1, and since $A_z \subsetneq A$ it follows that $u \leq_\gamma v' <_\gamma v$. Moreover, since $A_z \subseteq W_{\langle s_1 \rangle}$ it follows from Proposition 3.2.5 that $v' = \bigvee_\gamma A_z \in W_{\langle s_1 \rangle}$, and thus $s_1 \not\leq_\gamma v'$. Thus A_z is the set of atoms of $[u,v']_\gamma$, which implies that this interval is nuclear. It remains to show that $v' \lessdot_\gamma v$. It follows from $u \leq_\gamma v'$ and from the associativity of $\vee_\gamma$ that

$$v = \bigvee\nolimits_\gamma A = z \vee_\gamma \left(\bigvee\nolimits_\gamma A_z\right) = z \vee_\gamma v' = (s_1 \vee_\gamma u) \vee_\gamma v' = s_1 \vee_\gamma (u \vee_\gamma v') = s_1 \vee_\gamma v'.$$

Since $s_1 \not\leq_\gamma v'$ we can apply Lemma 3.3.7, and we obtain $v' \lessdot_\gamma s_1 \vee_\gamma v' = v$.

$(ii) \Rightarrow (i)$ Suppose now that there exists an element $v' \in [u,v]_\gamma$ satisfying $s_1 \not\leq_\gamma v' \lessdot_\gamma v$, and suppose that the interval $[u,v']_\gamma$ is nuclear. Let A' denote the set of atoms of $[u,v']_\gamma$. It follows from $s_1 \not\leq_\gamma v'$ that $z \notin A'$ and hence $A' \subseteq A_z$. Since $s_1 \leq_\gamma v$ and $v' \leq_\gamma v$ it follows from Lemma 3.3.7 that $s_1 \vee_\gamma v' = v$. Now let $z' \in A \setminus A'$. It follows from $z' \leq_\gamma v$ and from the associativity of $\vee_\gamma$ that

$$\bigvee\nolimits_\gamma (A' \cup \{z,z'\}) = \left(\bigvee\nolimits_\gamma A' \vee_\gamma z\right) \vee_\gamma z' = (v' \vee_\gamma z) \vee_\gamma z' = v \vee_\gamma z' = v,$$

and hence $v = \bigvee_\gamma A$, which implies that $[u,v]_\gamma$ is nuclear. $\square$

In particular, the element v' in Lemma 3.3.9(ii) is uniquely determined. We conclude this section by stating that for a finite Coxeter group W the cardinality of C_γ is given by the corresponding W-Catalan number.

Theorem 3.3.10 ([**94**, Theorem 9.1])

For any finite Coxeter group and any Coxeter element $\gamma \in W$ we have $|C_\gamma| = Cat(W)$.

3.4. Topological Properties of Closed Intervals of $\mathcal{C}_\gamma$

Let us now investigate the topology of closed intervals of $\mathcal{C}_\gamma$, analogously to the study of the m-Tamari lattices in Section 2.3. In particular, we prove a result analogous to Theorem 2.3.1, which implies that both generalizations of the Tamari lattices occurring in this thesis are natural in the sense that they generalize the topological properties of the Tamari lattices nicely.

Theorem 3.4.1

For any Coxeter group W and any Coxeter element $\gamma \in W$ every closed interval of $\mathcal{C}_\gamma$ is EL-shellable. Moreover, the Möbius function of $\mathcal{C}_\gamma$ takes values only in $\{-1,0,1\}$.

Before we actually prove Theorem 3.4.1, let us recall some special cases that were previously known in the literature. For finite Coxeter groups, the result on the Möbius function (or equivalently on the topological structure of the closed intervals) was already observed by Reading while he investigated so-called fan posets of certain hyperplane arrangements, see [**92**, Theorem 1.1].

The EL-shellability of the Cambrian lattices has first been investigated by Thomas in [**121**], where he showed that they are trim if associated with Coxeter groups of type A or of type B. (See Section 1.1.5 for the definition.) Later, Ingalls and Thomas showed that all Cambrian lattices associated with Weyl groups are trim, see [**64**, Theorem 4.17]. Since trim lattices are by definition left-modular, and left-modular lattices are EL-shellable, see

Theorem 1.1.22, the first part of Theorem 3.4.1 follows in the case of Weyl groups as a corollary. Their result was obtained by realizing the Cambrian lattices associated with a Weyl group as a hierarchy on torsion classes of representations of certain quivers (namely the orientations of the corresponding Coxeter diagram). The interested reader is referred to [64] for the details, or to THOMAS' chapter in [86], where the realization of the Tamari lattice of parameter $n+1$ as a hierarchy on torsion classes of representations of the directed path of length n is described.

Triggered by our results on the topology of the γ-Cambrian semilattices, PILAUD and STUMP proved that the *increasing flip poset* of any subword complex of a finite Coxeter group is EL-shellable, see [91, Theorem 4.2]. Moreover, each Cambrian lattice associated with a finite Coxeter group is the flip poset of some subword complex, see [90, Corollary 6.31]. Hence they recovered the first part of Theorem 3.4.1 for the finite Coxeter groups in a more general setting.

Summarizing the previous results, we obtain the following special case of Theorem 3.4.1.

THEOREM 3.4.2 ([64,91,92])

Let W be a finite Coxeter group, and let $\gamma \in W$ be a Coxeter element. Then, the lattice $\mathcal{C}_\gamma$ is EL-shellable, and it is trim when W is a Weyl group. Moreover, the Möbius function of $\mathcal{C}_\gamma$ takes values only in $\{-1,0,1\}$.

As described in the previous paragraph, this theorem was proven by using many different approaches, each of these approaches results from the research of a more general setting, and each of these approaches yields a special case of Theorem 3.4.1 as a corollary. In this section, we prove Theorem 3.4.1 uniformly, i.e. simultaneously for all Coxeter groups (finite or infinite) and all Coxeter elements.

3.4.1. EL-Shellability. In the remainder of this chapter, let (W,S) be a Coxeter system of rank n, let $S=\{s_1,s_2,\ldots,s_n\}$, and let $\gamma = s_1s_2\cdots s_n \in W$ be a (fixed) Coxeter element. Whenever we speak of the rank of W, then we actually mean the rank of (W,S). Whenever we speak of the rank of a parabolic subgroup W' of W, then we mean the rank of (W_J,J), where $J\subseteq S$ and W_J is conjugate to W'.

Recall from the definition that the cover relations in $\mathcal{W}$ are uniquely determined by a simple reflection, i.e. if $w,w'\in W$ satisfy $w \lessdot_S w'$, then there exists some $s\in S$ with $w'=ws$. The same does not necessarily hold for $\mathcal{C}_\gamma$. If $u,v\in C_\gamma$ with $u\lessdot_\gamma v$, then there exists a chain $u=w_0\lessdot_S w_1 \lessdot_S \cdots \lessdot_S w_k = v$ with $\pi_\downarrow^\gamma(w_i)=u$ for all $i\in\{0,1,\ldots,k-1\}$ and in general $k>1$. However, recall that every $w\in W$ has a unique γ-sorting word, and this word can be written as a subword of γ^∞ in the following form:

$$\gamma(w) = s_1^{\delta_{1,1}}s_2^{\delta_{1,2}}\cdots s_n^{\delta_{1,n}}|s_1^{\delta_{2,1}}s_2^{\delta_{2,2}}\cdots s_n^{\delta_{2,n}}|\cdots|s_1^{\delta_{l,1}}s_2^{\delta_{l,2}}\cdots s_n^{\delta_{l,n}},$$

for some $l\in\mathbb{N}$ with $\delta_{i,j}\in\{0,1\}, i\in\{1,2,\ldots,n\}$, and $j\in\{1,2,\ldots,l\}$, see also (3.2). Now we define the *set of filled positions of $\gamma(w)$* by

$$\alpha_\gamma(w) = \big\{(i-1)n+j \mid \delta_{i,j}=1\big\}\subseteq\mathbb{N}. \tag{3.6}$$

(Recall that $\gamma(w)$ denotes the γ-sorting word of w.) It is immediately clear that $\alpha_\gamma(w)$ depends on the reduced decomposition of γ, even though the γ-sortability of w does not. However, since we have chosen the fixed reduced decomposition $\gamma = s_1s_2\cdots s_n$, this will not be an issue. The following lemma is immediate.

Lemma 3.4.3

If $u,v \in W$ with $u \leq_S v$, then $\alpha_\gamma(u) \subseteq \alpha_\gamma(v)$.

Proof. It is another equivalent definition of the weak order that $u \leq_S v$ if and only if there exists a reduced decomposition of u which is a prefix of a reduced decomposition of v, see [23, Proposition 3.1.2(iv)]. It follows that any letter appearing in $\gamma(u)$ has to appear in $\gamma(v)$ as well. This is precisely the claim of this lemma. $\square$

Example 3.4.4

Let $W = A_3$ be generated by $S = \{s_1, s_2, s_3\}$ with $(s_1s_2)^3 = (s_1s_3)^2 = (s_2s_3)^3 = s_1^2 = s_2^2 = s_3^2 = \varepsilon$. Let $\gamma = s_1s_2s_3$. The corresponding lattice $\mathcal{C}_\gamma$ is shown in Figure 31. If $u = s_1s_2s_3|s_2$ and $v = s_2s_3|s_2|s_1$, then we have $\alpha_\gamma(u) = \{1,2,3,5\}$ and $\alpha_\gamma(v) = \{2,3,5,7\}$. It follows that $u \in C_\gamma$ and $v \notin C_\gamma$.

We see immediately that $w \in C_\gamma$ if and only if the following holds: if $i \in \alpha_\gamma(w)$ and $i > n$, then $i - n \in \alpha_\gamma(w)$. We notice that $\alpha_\gamma(u)$ contains both 5 and 2, while $\alpha_\gamma(v)$ does not contain $7 - 3 = 4$.

Now let us define our edge-labeling of $\mathcal{C}_\gamma$:

$$\lambda_\gamma : \mathcal{E}(\mathcal{C}_\gamma) \to \mathbb{N}, \quad (u,v) \mapsto \min\{i \mid i \in \alpha_\gamma(v) \setminus \alpha_\gamma(u)\}. \tag{3.7}$$

Since for $w \in W$ the set $\alpha_\gamma(w)$ depends on a particular choice of a reduced decomposition of γ so does λ_γ. However, since we focus on a fixed reduced decomposition of γ no problems will occur. See Figures 31 and 32 for examples of this labeling. The following properties of λ_γ are immediate.

Lemma 3.4.5

Let $\gamma = s_1s_2\cdots s_n$, and let $u,v \in C_\gamma$ with $u \leq_\gamma v$. If $i_0 = \min\{i \mid i \in \alpha_\gamma(v) \setminus \alpha_\gamma(u)\}$, then the following hold.

(i) The label i_0 appears in every maximal chain in $[u,v]_\gamma$.
(ii) The labels of a maximal chain in $[u,v]_\gamma$ are distinct.

Proof. (i) Suppose that this is not the case. Then there exists a maximal chain $u = w_0 \lessdot_\gamma w_1 \lessdot_\gamma \cdots \lessdot_\gamma w_k = v$ with $\lambda(w_i, w_{i+1}) \neq i_0$ for all $i \in \{0,1,\ldots,k-1\}$. This implies, however, that $i_0 \in \alpha_\gamma(u)$ if and only if $i_0 \in \alpha_\gamma(v)$, contradicting the definition of i_0.

(ii) Let $u = w_0 \lessdot_\gamma w_1 \lessdot_\gamma \cdots \lessdot_\gamma w_k = v$ be a maximal chain in $[u,v]_\gamma$, and assume that there are $i,j \in \{0,1,\ldots,k-1\}$ with $i < j$ and $\lambda(w_i,w_{i+1}) = s = \lambda(w_j,w_{j+1})$. By definition we have $s \in \alpha_\gamma(w_{i+1})$ and $s \notin \alpha_\gamma(w_j)$. Since $w_{i+1} \leq_S w_j$ Lemma 3.4.3 implies $\alpha_\gamma(w_{i+1}) \subseteq \alpha_\gamma(w_j)$, which is a contradiction. $\square$

We prove Theorem 3.4.1 by showing that λ_γ is an EL-labeling for every closed interval of $\mathcal{C}_\gamma$, and this proof will use induction on rank and length. Thus we need to understand how λ_γ behaves with respect to the recursive structure of the γ-sortable elements of W described in Proposition 3.2.6.

Lemma 3.4.6

Let $\gamma = s_1 s_2 \cdots s_n$, and let $u, v \in C_\gamma$ with $u \lessdot_\gamma v$. We have

$$\lambda_\gamma(u,v) = \begin{cases} 1, & \text{if } s_1 \not\leq_S u \text{ and } s_1 \leq_S v, \\ \lambda_{s_1\gamma s_1}(s_1 u, s_1 v) + 1, & \text{if } s_1 \leq_S u, \\ \lambda_{s_1\gamma}(u_{\langle s_1\rangle}, v_{\langle s_1\rangle}) + k, & \text{if } s_1 \not\leq_S v \text{ and } u \text{ and } v \text{ first differ in their } k\text{-th block.} \end{cases}$$

Proof. First let $s_1 \not\leq_S u$ and $s_1 \leq_S v$. By definition, the letter s_1 does not occur in $\gamma(u)$, but it does occur in $\gamma(v)$. Moreover, if s_1 occurs in $\gamma(v)$, then it has to occur in the first block of v since v is γ-sortable. Thus $\lambda_\gamma(u,v) = 1$.

Now let $s_1 \leq_S u$. By transitivity we have $s_1 \leq_S v$. It follows from Proposition 3.2.6 that $s_1 u$ and $s_1 v$ are both $s_1\gamma s_1$-sortable and that we have $s_1 u \lessdot_{s_1\gamma s_1} s_1 v$. Say that $\lambda_{s_1\gamma s_1}(s_1 u, s_1 v) = k$. By construction, $s_1\gamma s_1(u)$ is precisely the subword of $\gamma(u)$ starting at the second position. Thus if we consider $s_1\gamma s_1(s_1 u)$ as a subword of γ^∞, then we notice that the first position is empty, and likewise for $s_1 v$. If the first position of $(s_1\gamma s_1)^\infty$ where $s_1 u$ and $s_1 v$ differ is k, then the first position of γ^∞ where u and v differ is $k+1$. Hence $\lambda_\gamma(u,v) = \lambda_{s_1\gamma s_1}(s_1 u, s_1 v) + 1$.

Finally let $s_1 \not\leq_S v$. Again by transitivity we have $s_1 \not\leq_S u$, and Proposition 3.2.6 implies that $u = u_{\langle s_1\rangle}$ and $v = v_{\langle s_1\rangle}$ are $s_1\gamma$-sortable elements of $W_{\langle s_1\rangle}$. Again we have $s_1 u \lessdot_{s_1\gamma} s_1 v$. Say that the first position filled in $v_{\langle s_1\rangle}$ but not in $u_{\langle s_1\rangle}$ is in the k-th block of $v_{\langle s_1\rangle}$. If we consider $u_{\langle s_1\rangle}$ and $v_{\langle s_1\rangle}$ as subwords of γ^∞, then we have to add the letter s_1 with exponent 0 to each block of $u_{\langle s_1\rangle}$ and $v_{\langle s_1\rangle}$. Since the first difference of $u_{\langle s_1\rangle}$ and $v_{\langle s_1\rangle}$ is in the k-th block, the first difference of u and v is still in the k-th block, but each block has an additional first letter. Hence $\lambda_\gamma(u,v) = \lambda_{s_1\gamma}(u_{\langle s_1\rangle}, v_{\langle s_1\rangle}) + k$. □

Example 3.4.7

Let $W = B_3$ be generated by $S = \{s_1, s_2, s_3\}$ satisfying $(s_1s_2)^3 = (s_1s_3)^2 = (s_2s_3)^4 = s_1^2 = s_2^2 = s_3^2 = \varepsilon$, and let $\gamma = s_1s_2s_3$ be a Coxeter element of B_3. The corresponding lattice $\mathcal{C}_\gamma$ is shown in Figure 32.

First consider $u_1 = s_2s_3|s_2s_3$ and $v_1 = s_1s_2s_3|s_1s_2s_3|s_1s_2s_3$. It follows immediately that $\lambda_\gamma(u_1, v_1) = 1$.

Now consider $u_2 = s_1s_2s_3|s_1s_2$ and $v_2 = s_1s_2s_3|s_1s_2s_3$. Then $s_1u_2 = s_2s_3s_1|s_2$ and $s_1v_2 = s_2s_3s_1|s_2s_3$ considered as $s_1\gamma s_1$-sorting words. We have

$$\lambda_{s_1\gamma s_1}(s_1u_2, s_1v_2) = 5 \quad \text{and} \quad \lambda_\gamma(u_2, v_2) = 6.$$

Finally consider $u_3 = s_2s_3|s_2$ and $v_3 = s_2s_3|s_2s_3$. The $(s_1\gamma)$-sorting words of $(u_3)_{\langle s_1\rangle}$ and $(v_3)_{\langle s_1\rangle}$ written as in (3.2) are

$$(u_3)_{\langle s_1\rangle} = s_2^1 s_3^1 | s_2^1 s_3^0 \quad \text{and} \quad (v_3)_{\langle s_1\rangle} = s_2^1 s_3^1 | s_2^1 s_3^1,$$

and the corresponding γ-sorting words are

$$(u_3)_{\langle s_1\rangle} = s_1^0 s_2^1 s_3^1 | s_1^0 s_2^1 s_3^0 \quad \text{and} \quad (v_3)_{\langle s_1\rangle} = s_1^0 s_2^1 s_3^1 | s_1^0 s_2^1 s_3^1.$$

Hence $\lambda_{s_1\gamma}(u_3)_{\langle s_1\rangle}, v_3)_{\langle s_1\rangle}) = 4$ and $\lambda_\gamma(u_3, v_3) = 6$.

Now we are ready to prove the first part of Theorem 3.4.1, and we state this as a separate theorem.

Theorem 3.4.8

Let (W,S) be a Coxeter system of rank n, where $S=\{s_1,s_2,\ldots,s_n\}$, and let $\gamma=s_1s_2\cdots s_n\in W$ be a Coxeter element. The labeling λ_γ defined in (3.7) is an EL-labeling of every closed interval of $\mathcal{C}_\gamma$.

Proof. Let $u,v\in C_\gamma$ with $u\leq_\gamma v$. We need to show that there exists a unique rising maximal chain in $[u,v]_\gamma$, and that this chain is lexicographically first. We proceed by induction on rank and length. If W has rank 2 or if $\ell_S(v)\leq 2$, then the claim is trivially true. Hence let W have rank n and let $\ell_S(v)=k$. Suppose that the claim is true for all parabolic subgroups of W of rank $<n$ and for all closed intervals $[u',v']_{\gamma'}$ for some Coxeter element $\gamma'\in W$, where $\ell_S(v')<k$. We distinguish three cases:

(i) Let $s_1\leq_\gamma u$. Then, s_1 is the first letter of every γ-sorting word of every element in $[u,v]_\gamma$, and Lemma 3.3.8 implies that $[u,v]_\gamma\cong[s_1u,s_1v]_{s_1\gamma s_1}$. Since $\ell_S(s_1v)=k-1$ the induction hypothesis implies the existence of a unique maximal rising chain $s_1u=s_1x_0\lessdot_{s_1\gamma s_1} s_1x_1\lessdot_{s_1\gamma s_1}\cdots\lessdot_{s_1\gamma s_1} s_1x_t=s_1v$ that is lexicographically first among all maximal chains in $[s_1u,s_1v]_{s_1\gamma s_1}$. Lemma 3.4.6 implies that the corresponding chain $u=x_0\lessdot_\gamma x_1\lessdot_\gamma\cdots\lessdot_\gamma x_t=v$ is the unique maximal rising chain in $[u,v]_\gamma$ and that it is lexicographically first among all maximal chains in $[u,v]_\gamma$.

(ii) Let $s_1\not\leq_\gamma u$ and $s_1\leq_\gamma v$. Lemma 3.3.7 implies that $z=s_1\vee_\gamma u$ covers u. Hence by Lemma 3.3.8 the interval $[z,v]_\gamma$ is isomorphic to the interval $[s_1z,s_1v]_{s_1\gamma s_1}$, and analogously to (i) we can find a unique rising maximal chain in $[z,v]_\gamma$ that is lexicographically first among all maximal chains in $[z,v]_\gamma$, say $z=x_1\lessdot_\gamma x_2\lessdot_\gamma\cdots\lessdot_\gamma x_t=v$. Lemma 3.4.6 implies now that $\lambda_\gamma(u,z)=1$ and $\lambda_\gamma(x_1,x_2)\geq 2$. Hence the chain $C:u=x_0\lessdot_\gamma x_1\lessdot_\gamma\cdots\lessdot_\gamma x_t=v$ is a rising maximal chain in $[u,v]_\gamma$. Now suppose that there is another element $z'\in C_\gamma$ with $u\lessdot_\gamma z'\leq_\gamma v$ and $\lambda_\gamma(u,z')=1$. By definition of λ_γ the letter s_1 has to occur in $\gamma(z')$, which implies $s_1\leq_\gamma z'$. Thus z' is an upper bound for both s_1 and u, and by the uniqueness of (existing) joins in $\mathcal{C}_\gamma$ it follows that $z\leq_\gamma z'$. Since both z and z' cover u it follows that $z=z'$. Hence C is the lexicographically first maximal chain in $[u,v]_\gamma$. Moreover, Lemma 3.4.5 implies that the label 1 occurs in every maximal chain in $[u,v]_\gamma$, which implies that C is the unique rising chain in $[u,v]_\gamma$.

(iii) Let $s_1\not\leq_\gamma v$. In this case no element of $[u,v]_\gamma$ contains the letter s_1 in its γ-sorting word. Lemma 3.3.8 implies that $[u,v]_\gamma$ is isomorphic to the interval $\big[u_{\langle s_1\rangle},v_{\langle s_1\rangle}\big]_{s_1\gamma}$ in $W_{\langle s_1\rangle}$. Since $W_{\langle s_1\rangle}$ is a parabolic subgroup of W of rank $n-1$, by induction hypothesis there exists a unique maximal rising chain $u_{\langle s_1\rangle}=(x_0)_{\langle s_1\rangle}\lessdot_{s_1\gamma}(x_1)_{\langle s_1\rangle}\lessdot_{s_1\gamma}\cdots\lessdot_{s_1\gamma}(x_t)_{\langle s_1\rangle}=v_{\langle s_1\rangle}$ that is lexicographically first among all maximal chains in $[u_{\langle s_1\rangle},v_{\langle s_1\rangle}]_{s_1\gamma}$. Let $i,j\in\{1,2,\ldots,t\}$ with $i<j$. Say that the first block where $(x_i)_{\langle s_1\rangle}$ and $(x_{i+1})_{\langle s_1\rangle}$ differ is their d_i-th block, and say that the first block where $(x_j)_{\langle s_1\rangle}$ and $(x_{j+1})_{\langle s_1\rangle}$ differ is their d_j-th block. Since $\lambda_{s_1\gamma}\big((x_i)_{\langle s_1\rangle},(x_{i+1})_{\langle s_1\rangle}\big)<\lambda_{s_1\gamma}\big((x_j)_{\langle s_1\rangle},(x_{j+1})_{\langle s_1\rangle}\big)$ it follows that $d_i\leq d_j$. Lemma 3.4.6 implies that the associated chain $u=x_0\lessdot_\gamma x_1\lessdot_\gamma\cdots\lessdot_\gamma x_t=v$ is the unique rising maximal chain in $[u,v]_\gamma$ and that it is lexicographically first among all maximal chains in $[u,v]_\gamma$. $\square$

The next result follows immediately from the proof of Theorem 3.4.8.

Corollary 3.4.9

Let $\gamma=s_1s_2\cdots s_n$, and let $w\in C_\gamma$ with $\ell_S(w)=k$. Then, $\ell\big([\varepsilon,w]_\gamma\big)=k$.

Proof. It follows from the EL-shellability of $[\varepsilon, w]_\gamma$ and Lemma 1.1.6 that $\ell\big([\varepsilon, w]_\gamma\big)$ is precisely the length of the unique rising maximal chain from ε to w. If $\gamma(w) = s_{i_1}s_{i_2}\cdots s_{i_k}$, then the chain $\varepsilon = x_0 \lessdot_\gamma x_1 \lessdot_\gamma \cdots \lessdot_\gamma x_k = w$, where $x_j = s_{i_1}s_{i_2}\cdots s_{i_j}$ for $j \in \{1, 2, \ldots, k\}$, is rising with respect to λ_γ, which implies the claim. $\square$

3.4.2. The Möbius Function. In this section we prove the second part of Theorem 3.4.1, namely that the Möbius function of $\mathcal{C}_\gamma$ takes only values in $\{-1, 0, 1\}$. Again we state this in a separate theorem.

Theorem 3.4.10

Let (W, S) be a Coxeter system of rank n, where $S = \{s_1, s_2, \ldots, s_n\}$, and let $\gamma = s_1 s_2 \cdots s_n \in W$ be a Coxeter element. If $u, v \in C_\gamma$ with $u \leq_\gamma v$, then $\mu_{\mathcal{C}_\gamma}(u, v) \in \{-1, 0, 1\}$.

Proof. In view of Proposition 1.1.14, it is sufficient to show that there exists at most one falling chain in $[u, v]_\gamma$, and again we proceed by induction on rank and length. Again we may assume that the rank of W is at least 3 and $\ell_S(v) \geq 3$, because the result is trivial otherwise. Hence let W have rank n, let $\ell_S(v) = k$, and suppose that the claim is true for all parabolic subgroups of W of rank $< n$, and for all closed intervals $[u', v']_{\gamma'}$ for some Coxeter element $\gamma' \in W$, where $\ell_S(v') < k$. We distinguish three cases:

(i) Let $s_1 \leq_\gamma u$. The result follows by induction on length, following the steps of case (i) in the proof of Theorem 3.4.8.

(ii) Let $s_1 \not\leq_\gamma u$ and $s_1 \leq_\gamma v$. It follows from Lemma 3.4.5(i) that the label 1 occurs in every maximal chain $u = x_0 \lessdot_\gamma x_1 \lessdot_\gamma \cdots \lessdot_\gamma x_t = v$. Hence such a chain can be falling only if $\lambda_\gamma(x_{t-1}, x_t) = 1$. If there is no element $v_1 \in [u, v]_\gamma$ with $v_1 \lessdot_\gamma v$ and $\lambda_\gamma(v_1, v) = 1$, then there is no falling maximal chain in $[u, v]_\gamma$, which implies $\mu_{\mathcal{C}_\gamma}(u, v) = 0$. Otherwise consider the interval $[u, v_1]_\gamma$. By the choice of v_1 it follows from Lemma 3.4.5(ii) that the label 1 does not occur in any maximal chain from u to v_1, and hence every maximal falling chain from u to v_1 can be extended to a maximal falling chain from u to v. Conversely every maximal falling chain from u to v can be restricted to a maximal falling chain from u to v_1. Since $\ell_S(v_1) = k - 1$, our induction hypothesis implies that there is at most one falling chain in $[u, v_1]$, which implies the claim.

(iii) Let $s_1 \not\leq_\gamma v$. The result follows by induction on rank, following the steps of case (iii) in the proof of Theorem 3.4.8. $\square$

Proof of Theorem 3.4.1. This follows from Theorems 3.4.8 and 3.4.10. $\square$

We complete this section with the characterization of the spherical intervals of $\mathcal{C}_\gamma$.

Theorem 3.4.11

Let (W, S) be a Coxeter system of rank n, and let $\gamma = s_1 s_2 \cdots s_n \in W$ be a Coxeter element. Further, let $u, v \in C_\gamma$ with $u \leq_\gamma v$, and let k denote the number of atoms in $[u, v]_\gamma$. Then,

$$\mu_{\mathcal{C}_\gamma}(u, v) = \begin{cases} (-1)^k, & \text{if } [u, v]_\gamma \text{ is nuclear,} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. In view of Proposition 1.1.14, we need to show that there exists a falling chain in $[u, v]_\gamma$ if and only if $[u, v]_\gamma$ is nuclear and that this chain has length k if and only if $[u, v]_\gamma$ has k atoms. Similar to the proof of Theorem 3.4.8, we proceed by induction on rank and length, and we may assume that the rank of W is at least 3 and $\ell_S(v) \geq 3$, because the result is trivial otherwise. Hence let W have rank n, let $\ell_S(v) = k$, and suppose that the claim is true for all parabolic subgroups of W of rank $< n$, and for all closed intervals $[u', v']_{\gamma'}$ for some Coxeter element $\gamma' \in W$, where $\ell_S(v') < k$. We distinguish three cases:

(i) Let $s_1 \leq_\gamma u$. The result follows by induction on length, following the steps of case (i) in the proof of Theorem 3.4.8.

(ii) Let $s_1 \not\leq_\gamma u$ and $s_1 \leq_\gamma v$. If $[u,v]_\gamma$ is nuclear, then Lemma 3.3.9 implies that there exists a unique element $v' \in C_\gamma$ with $u \leq_\gamma v' \lessdot_\gamma v$ such that $[u,v']_\gamma$ is nuclear and $s_1 \not\leq_\gamma v'$. By induction on rank it follows that there exists a maximal falling chain $C' : u = x_0 \lessdot_\gamma x_1 \lessdot_\gamma \cdots \lessdot_\gamma x_{k-1} = v'$ and that $[u,v']_\gamma$ has $k-1$ atoms. Lemma 3.4.5 implies that $1 \notin \lambda_\gamma(C')$, and Lemma 3.4.6 implies that $\lambda_\gamma(v',v) = 1$. Thus the chain $C : u = x_0 \lessdot_\gamma x_1 \lessdot_\gamma \cdots \lessdot_\gamma x_{t-1} \lessdot_\gamma x_k = v$ is a falling maximal chain in $[u,v]_\gamma$. Lemma 3.3.9 implies now that $[u,v]_\gamma$ has k atoms.
Conversely suppose that $C : u = x_0 \lessdot_\gamma x_1 \lessdot_\gamma \cdots \lessdot_\gamma x_k = v$ is a falling maximal chain in $[u,v]_\gamma$. Lemma 3.4.6(i) implies that $\lambda_\gamma(x_{t-1}, x_t) = 1$, which in turn implies that $s_1 \not\leq_\gamma x_{t-1}$. The restricted chain $C' : u = x_0 \lessdot_\gamma x_1 \lessdot_\gamma \cdots \lessdot_\gamma x_{k-1}$ is still falling, and by induction we see that $[u, x_{t-1}]_\gamma$ is a nuclear interval with $k-1$ atoms. Since $s_1 \not\leq_\gamma x_{k-1} \lessdot_\gamma v$, it follows from Lemma 3.3.9 that $[u,v]_\gamma$ is nuclear and has k atoms.

(iii) Let $s_1 \not\leq_\gamma v$. The result follows by induction on rank, following the steps of case (iii) in the proof of Theorem 3.4.8. $\square$

Example 3.4.12

Let $\tilde{A}_2$ be the infinite Coxeter group of rank 3, generated by $S = \{s_0, s_1, s_2\}$ with $(s_0s_1)^3 = (s_0s_2)^3 = (s_1s_2)^3 = s_0^2 = s_1^2 = s_2^2 = \varepsilon$, and consider the Coxeter element $\gamma = s_0s_1s_2$. Figure 33 shows the sub-semilattice of $\mathcal{C}_\gamma$, consisting of all elements $w \in C_\gamma$ with $\ell_S(w) \leq 7$.

Example 3.4.13

Let $\tilde{C}_3$ be the infinite Coxeter group of rank 4, generated by $S = \{s_0, s_1, s_2, s_3\}$ with $(s_0s_1)^4 = (s_0s_2)^2 = (s_0s_3)^2 = (s_1s_2)^3 = (s_1s_3)^2 = (s_2s_3)^4 = s_0^2 = s_1^2 = s_2^2 = s_3^2 = \varepsilon$, and consider the Coxeter element $\gamma = s_0s_1s_2s_3$. Figure 34 shows the interval $[\varepsilon, s_0s_1s_2s_3|s_1s_2s_3|s_1s_2s_3]_\gamma$ in $\mathcal{C}_\gamma$, together with the EL-labeling defined in (3.7).

3.4.3. Trimness. Recall from Theorem 3.4.2 that the Cambrian lattices associated with a Weyl group are trim lattices, see [64, Theorem 4.17]. We have computed the length of an interval $[\varepsilon, w]_\gamma$ in $\mathcal{C}_\gamma$ for arbitrary Coxeter groups, see Corollary 3.4.9. In this section we extend [64, Theorem 4.17] by showing that every closed interval of $\mathcal{C}_\gamma$ is in fact a trim lattice.

Theorem 3.4.14

Let (W,S) be a Coxeter system of rank n, where $S = \{s_1, s_2, \ldots, s_n\}$, and let $\gamma = s_1s_2\cdots s_n \in W$ be a Coxeter element. Every closed interval of $\mathcal{C}_\gamma$ is a trim lattice.

We prove Theorem 3.4.14 in several steps, starting with the computation of the cardinality of the join- and meet-irreducible elements. Subsequently we explicitly construct a left-modular chain.

Proposition 3.4.15

Let $\gamma = s_1s_2\cdots s_n$, and let $w \in C_\gamma$ with $\ell_S(w) = k$. Then $\Big|\mathcal{J}\big([\varepsilon,w]_\gamma\big)\Big| = k = \Big|\mathcal{M}\big([\varepsilon,w]_\gamma\big)\Big|$.

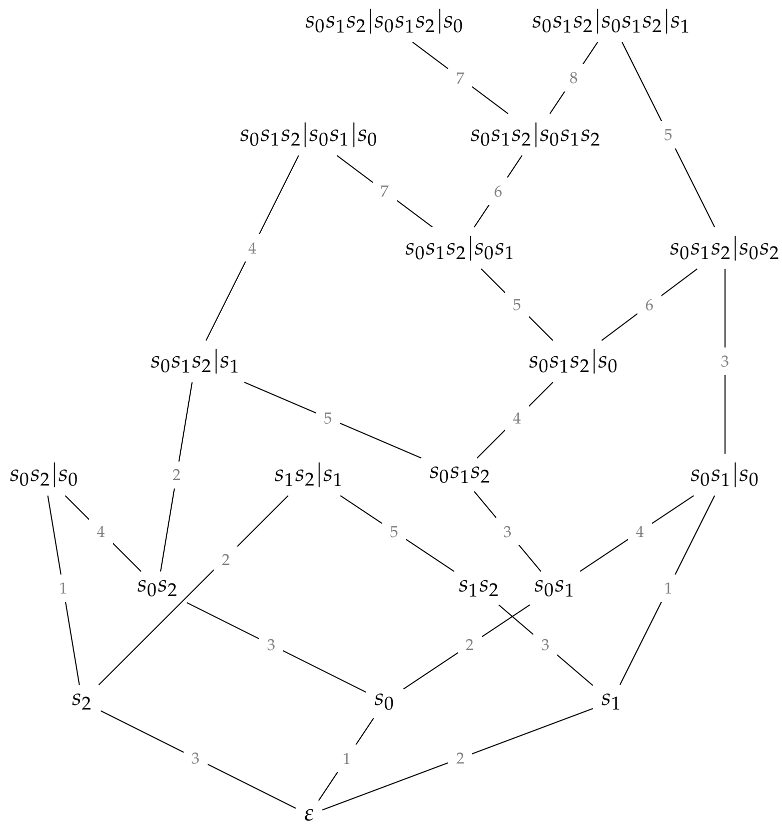


FIGURE 33. The first seven ranks of an $\tilde{A}_2$-Cambrian semilattice with the labeling defined in (3.7).

PROOF. First of all recall from Lemma 1.1.26 and Proposition 3.3.6 that $\left|\mathcal{J}\big([\varepsilon,w]_\gamma\big)\right| = \left|\mathcal{M}\big([\varepsilon,w]_\gamma\big)\right|$. We will show that $\left|\mathcal{M}\big([\varepsilon,w]_\gamma\big)\right| = \ell_S(w)$, and we proceed by induction on rank and length. If W has rank 2 or if $\ell_S(w) \leq 2$, then the result is trivially true. Hence let W have rank n, let $\ell_S(w) = k$, and suppose that the claim is true for all parabolic subgroups of W of rank $< n$, and for all γ'-sortable elements $w' \in W$ for some Coxeter element $\gamma' \in W$ with $\ell_S(w') < k$. We distinguish two cases:

(i) Let $s_1 \leq_\gamma w$. Lemma 3.3.8 implies that the interval $[s_1,w]_\gamma$ is isomorphic to the interval $[\varepsilon, s_1w]_{s_1\gamma s_1}$, and by induction on length, it follows that there are $k-1$ meet-irreducible elements in $[s_1,w]_\gamma$. Let $m \in \mathcal{M}\big([s_1,w]_\gamma\big)$, and suppose that $m \notin \mathcal{M}\big([\varepsilon,w]_\gamma\big)$. Thus there exist distinct elements $m_1, m_2 \in C_\gamma$ with $m \lessdot_\gamma m_1, m_2$ and $m_1, m_2 \leq_\gamma w$. Then, however, it follows from $s_1 \leq_\gamma m$ that $m_1, m_2 \in [s_1,w]_\gamma$, which contradicts $m \in \mathcal{M}\big([s_1,w]_\gamma\big)$. Hence every meet-irreducible element of $[s_1,w]_\gamma$ is also meet-irreducible in $[\varepsilon,w]_\gamma$. Now we show that there is exactly one additional meet-irreducible element in $[\varepsilon,w]_\gamma$, denoted by z. Let $a_1, a_2, \ldots, a_t$ denote the atoms of $[\varepsilon,w]_\gamma$ that are different from s_1, and let that $x = a_1 \vee_\gamma a_2 \vee_\gamma \cdots \vee_\gamma a_t$. If we set $x' = s_1 \vee_\gamma x$, then Lemma 3.3.7 implies that $x \lessdot_\gamma x' \leq_\gamma w$. If $x' = w$, then x is meet-irreducible in $[\varepsilon,w]_\gamma$, and we set $z = x$. Otherwise, suppose that we can find two distinct elements $y_1, y_2 \leq_\gamma w$ that are different from x' and that satisfy $x \lessdot_\gamma y_1, y_2$. If we set $y_1' = s_1 \vee_\gamma y_1$ and $y_2' = s_1 \vee_\gamma y_2$, then we have $x' \leq_\gamma y_1', y_2'$ and the sublattice of $[\varepsilon,w]_\gamma$ induced

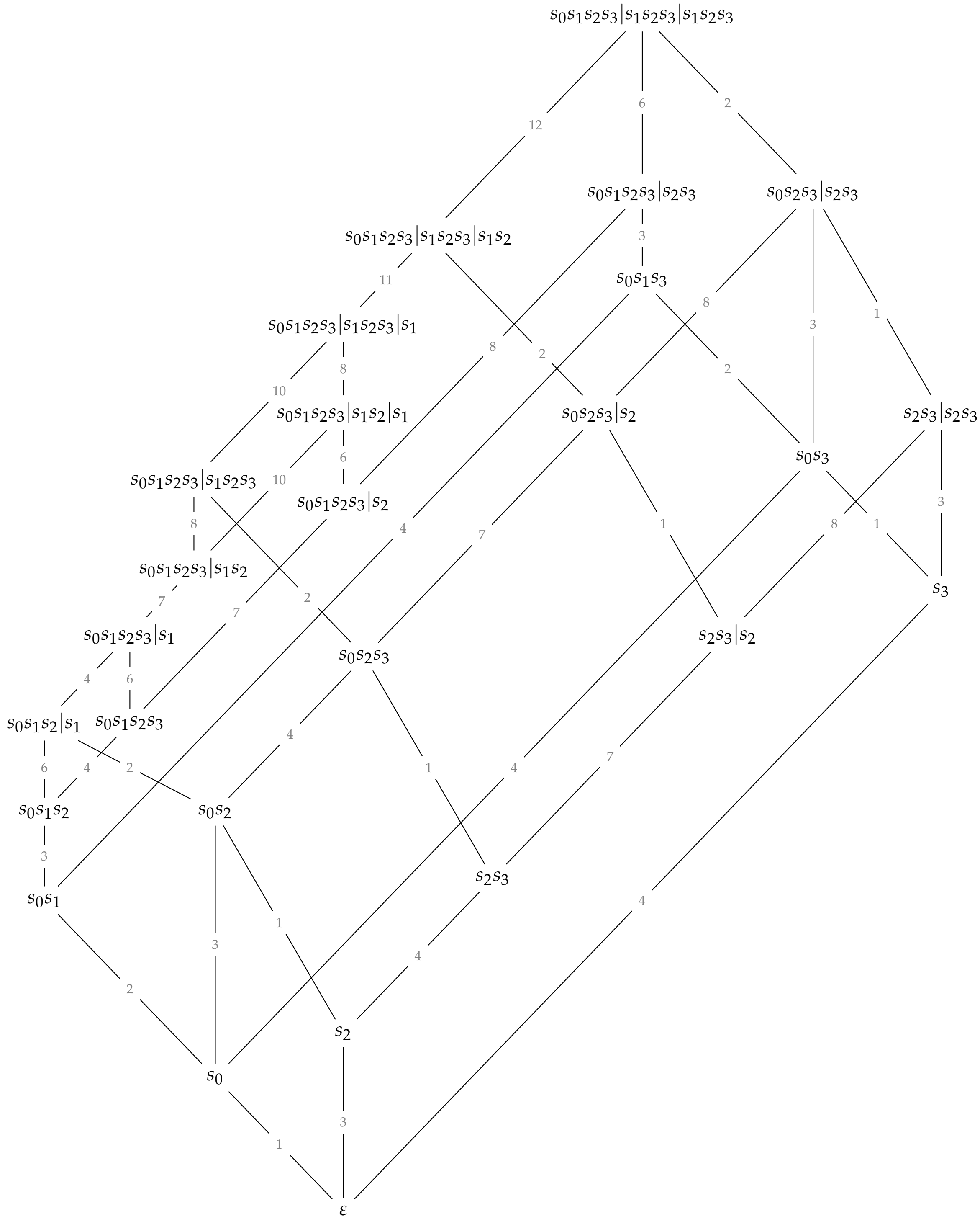


FIGURE 34. An interval of a $\tilde{C}_3$-Cambrian semilattice with the labeling defined in (3.7).

by the set $\{x, x', y_1, y_2, y_1', y_2', y_1' \vee_\gamma y_2'\}$ forms the dual of the lattice $\mathcal{L}_4$ from Figure 3(d), contradicting the semidistributivity of $\mathcal{C}_\gamma$. Hence x can have at most one upper cover other than x'. If this is not the case, then x is meet-irreducible in $[\varepsilon, w]_\gamma$ and we set $z = x$. Otherwise, denote this upper cover by y. Now we can iterate the previous reasoning with y instead of x exactly analogously, and since $[\varepsilon, w]_\gamma$ is finite, we eventually find the desired meet-irreducible element z. If u is some element in $[\varepsilon, w]_\gamma$ with $s_1 \not\leq_\gamma u <_\gamma z$, then we have $u <_\gamma z$ and $u \lessdot_\gamma s_1 \vee_\gamma u \not\leq_\gamma z$, and it follows that $u \notin \mathcal{M}\big([\varepsilon, w]_\gamma\big)$. Finally, every element that lies strictly between z and w was already considered in the induction step, since it has to be above s_1. Thus we have exactly k meet-irreducible elements in $[\varepsilon, w]_\gamma$.

(ii) Let $s_1 \not\leq_\gamma w$. Lemma 3.3.8 implies $[\varepsilon, w]_\gamma \cong \big[\varepsilon, w_{\langle s_1\rangle}\big]_{s_1\gamma}$, and the result follows by induction on rank. □

Corollary 3.4.16

Let $\gamma = s_1 s_2 \cdots s_n$, and let $w \in C_\gamma$. Then, the interval $[\varepsilon, w]_\gamma$ is extremal.

Proof. This follows from Corollary 3.4.9 and Lemma 3.4.15. □

Proposition 3.4.17

Let $\gamma = s_1 s_2 \cdots s_n$, and let $w \in C_\gamma$. Then, the interval $[\varepsilon, w]_\gamma$ is left-modular.

Proof. Let $\gamma(w) = s_{i_1} s_{i_2} \cdots s_{i_k}$ be the γ-sorting word of w, and define $x_j = s_{i_1} s_{i_2} \cdots s_{i_j}$ for $j \in \{1, 2, \ldots, k\}$. We show now that the chain $\varepsilon \lessdot_\gamma x_1 \lessdot_\gamma x_2 \lessdot_\gamma \cdots \lessdot_\gamma x_k = w$ is left-modular. Let $j \in \{1, 2, \ldots, k\}$, and let $y, y' \in C_\gamma$ with $y \lessdot_\gamma z \leq_\gamma w$. In view of Theorem 1.1.21, it suffices to show that

$$\text{either} \quad x_j \vee_\gamma y = x_j \vee_\gamma z \quad \text{or} \quad x_j \wedge_\gamma y = x_j \wedge_\gamma z, \tag{3.8}$$

and again we proceed by induction on rank and length. If W has rank 2 or if $\ell_S(w) = 2$, then the result is trivially true. Hence let W have rank n and let $\ell_S(w) = k$. Suppose that the claim is true for all parabolic subgroups of W of rank $< n$, and for all γ'-sortable elements $w' \in W$ for some Coxeter element $\gamma' \in W$ with $\ell_S(w') < k$. We distinguish two cases:

(i) Let $s_1 \leq_\gamma w$. In particular we have $x_1 = s_1$. Moreover, Lemma 3.3.8 implies that $[s_1, w]_\gamma$ is isomorphic to $[\varepsilon, s_1 w]_{s_1\gamma s_1}$. If $s_1 \leq_\gamma y$, then (3.8) is satisfied by induction hypothesis.

If $s_1 \not\leq_\gamma y$ but $s_1 \leq_\gamma z$, then Lemma 3.3.7 implies $z = s_1 \vee_\gamma y$, which in turn implies $x_j \vee_\gamma z = x_j \vee_\gamma (s_1 \vee_\gamma y) = x_j \vee_\gamma y$. Now suppose that $x_j \wedge_\gamma y = x_j \wedge_\gamma z$. Since $s_1 \leq_\gamma x_j, z$ it follows that $s_1 \leq_\gamma x_i \wedge_\gamma z = x_i \wedge_\gamma y \leq_\gamma y$, which is a contradiction. Hence (3.8) is satisfied.

If $s_1 \not\leq_\gamma z$, then define $y' = s_1 \vee_\gamma y$ and $z' = s_1 \vee_\gamma z$. Lemma 3.3.7 implies $y \lessdot_\gamma y'$ and $z \lessdot_\gamma z'$ and hence $y' <_\gamma z'$. Both s_1 and $x_j \wedge_\gamma y$ are lower bounds for x_j and y', which implies that either $s_1 \leq_\gamma x_j \wedge_\gamma y$ or $x_j \wedge_\gamma y \leq_\gamma s_1$. If $s_1 \leq_\gamma x_j \wedge_\gamma y$, then it follows that $s_1 \leq_\gamma y$, which is a contradiction. Since s_1 is an atom of $[\varepsilon, w]_\gamma$, we conclude thus that $x_j \wedge_\gamma y = \varepsilon$. The same reasoning implies $x_j \wedge_\gamma z = \varepsilon$. By induction hypothesis and Lemma 3.3.8 we conclude that x_j is left-modular in the interval $[s_1, w]_\gamma$, and by definition we have $(y' \vee_\gamma x_j) \wedge_\gamma z' = y' \vee_\gamma (x_j \wedge_\gamma z')$. Suppose that $x_j \vee_\gamma y = x_j \vee_\gamma z$. This implies $x_j \vee_\gamma y' = x_j \vee_\gamma z'$. Further we obtain

$$y' \vee_\gamma z = z' = (z' \vee_\gamma x_j) \wedge_\gamma z' = (y' \vee_\gamma x_j) \wedge_\gamma z' = y' \vee (x_j \wedge_\gamma z'), \quad \text{but}$$
$$y' \vee_\gamma (x_j \wedge_\gamma z \wedge_\gamma z') = y' \vee_\gamma (x_j \wedge_\gamma z) = y' \vee_\gamma (x_j \wedge_\gamma y) \leq_\gamma y' \vee_\gamma (x_j \wedge_\gamma y') = y' <_\gamma z',$$

which contradicts the semidistributivity of $[\varepsilon, w]_\gamma$. Hence (3.8) is satisfied.

(ii) Let $s_1 \not\leq_\gamma w$. Then, Lemma 3.3.8 implies $[\varepsilon, w]_\gamma \cong [\varepsilon, w_{\langle s_1\rangle}]_{s_1\gamma}$, and the result follows by induction on rank. $\square$

Proof of Theorem 3.4.14. Corollary 3.4.16 and Proposition 3.4.17 imply that for every $w \in C_\gamma$, the interval $[\varepsilon, w]_\gamma$ is a trim lattice. Clearly, for $u, v \in C_\gamma$, the interval $[u, v]_\gamma$ is a subinterval of $[\varepsilon, v]_\gamma$, and Theorem 1.1.23 implies that intervals of trim lattices are trim again, which completes the proof. $\square$

3.5. Structural Properties of Closed Intervals of $\mathcal{C}_\gamma$

We conclude our study of the γ-Cambrian semilattices with a brief structural investigation, and start with a few historical remarks. The Tamari lattices play an important role in lattice theory, since they possess a wealth of nice structural properties. In [125, Corollary, page 55], Urquhart proved that the Tamari lattices are so-called split lattices in the sense of [78], and it follows then from [78, Theorem 5.1] that they are bounded-homomorphic images of free lattices, and hence semidistributive. Moreover, every distributive lattice can be embedded into a Tamari lattice, see [76, Corollary, page 288]. It was conjectured by Geyer that every bounded-homomorphic image of a free lattice is a sublattice of some Tamari lattice, see [58, Conjecture 3.6]. This conjecture was disproven only recently, when Santocanale and Wehrung introduced an infinite collection of lattice-theoretic identities, for which they showed that they hold in the Tamari lattices, but not in every finite bounded-homomorphic image of a free lattice, see [102]. In fact, already the weak order lattice of A_3 and the two Cambrian lattices of A_3 that are not isomorphic to $\mathcal{T}_4$ provide counterexamples. The fact that the weak order of a finite Coxeter group is a bounded-homomorphic image of a free lattice is due to Caspard, Le Conte de Poly-Barbut and Morvan, see [37, Theorem 6]. Since every sublattice of a bounded-homomorphic image of a free lattice is again a bounded-homomorphic image of a free lattice, see [55, Corollary 2.17], it follows by definition that the Cambrian lattices associated with a finite Coxeter group have this property as well.

3.5.1. Bounded-Homomorphic Images of Free Lattices. In fact, Caspard et. al. showed that the weak order on a finite Coxeter group belongs to the class of $\mathcal{HH}$-lattices, which is a subclass of the class of bounded lattices, see Definition 3.5.4 below. We remark that the results on the weak order obtained in [37] extend straightforwardly to closed intervals of the weak order of an arbitrary, possibly infinite Coxeter group. We extend their ideas to prove that every closed interval of a γ-Cambrian semilattice is an $\mathcal{HH}$-lattice, where $\gamma \in W$ is a Coxeter element of some Coxeter group W, a result that cannot be deduced from the fact that the weak order is an $\mathcal{HH}$-lattice. It was proven explicitly in [36] that the Tamari lattices are $\mathcal{HH}$-lattices as well, an observation that motivated the research presented in this section. In particular, we prove the following theorem.

Theorem 3.5.1

Let (W, S) be a Coxeter system of rank n, where $S = \{s_1, s_2, \ldots, s_n\}$, and let $\gamma = s_1 s_2 \cdots s_n$. Every closed interval of $\mathcal{C}_\gamma$ is an $\mathcal{HH}$-lattice, and therefore a bounded-homomorphic image of a free lattice.

Recall from Section 1.1.4 that an interval $[p, q]$ in a finite lattice $\mathcal{P} = (P, \leq)$ is called a *2-facet* if its proper part is the disjoint union of two chains. It is immediate that in this case there are two triples (p_1, p, p_2) and (q_1, q, q_2) associated with $[p, q]$ that satisfy $p \lessdot p_1, p_2$ and $q_1, q_2 \lessdot q$. These triples are called an *anti-hat* and a *hat of $\mathcal{P}$*, and they are denoted by $\mathrm{V}(p_1, p, p_2)$ and

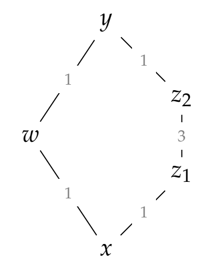


FIGURE 35. The pentagon lattice is a 2-facet. The associated hat is $\Lambda(w, y, z_2)$, and the associated anti-hat is $\mathrm{V}(w, x, z_1)$. Its Hasse diagram is labeled by a 2-facet labeling, and the identity map is a 2-facet rank function with respect to this labeling.

$\Lambda(q_1, q, q_2)$, respectively. Recall the definition of a 2-facet labeling from Definition 1.1.18 on page 14.

DEFINITION 3.5.2

Let $\mathcal{P} = (P, \leq)$ be a lattice, and let λ be a 2-facet labeling of $\mathcal{P}$. A function $r : P \to \mathbb{N}$ is called a *2-facet rank function of $\mathcal{P}$ with respect to λ* if it satisfies the following property: for every 2-facet $[p, q]$ of $\mathcal{P}$ with maximal chains C and C', where their corresponding label sequences are $\lambda(C) = (t_1, t_2, \ldots, t_k)$ and $\lambda(C') = (t'_1, t'_2, \ldots, t'_{k'})$, we have

$$r(t_1), r(t_k) < r(t_2), r(t_{k-1}) < r(t_3), \ldots, r(t_{(k+1)/2+1}) < r(t_{(k+1)/2})$$

if k is odd, and

$$r(t_1), r(t_k) < r(t_2), r(t_{k-1}) < r(t_3), \ldots, r(t_{k/2+2}) < r(t_{k/2}), r(t_{k/2+1})$$

if k is even, and likewise for $\lambda(C')$.

EXAMPLE 3.5.3

Figure 35 shows the pentagon lattice, and the edges of its Hasse diagram are labeled by a 2-facet labeling and the identity map is a 2-facet rank function with respect to this lattice. Further examples can be seen in Figures 36 and 37.

DEFINITION 3.5.4 ([37, Definition 10])

A lattice $\mathcal{P}$ is called an *$\mathcal{HH}$-lattice* if and only if it satisfies the following conditions:

(H1) $\mathcal{P}$ is finite and semidistributive;

(H2) for every anti-hat $\mathrm{V}(p_1, p, p_2)$ there exists a unique hat $\Lambda(q_1, q, q_2)$ with $q = p_1 \vee p_2$ such that $[p, q]$ is a 2-facet;

(H3) for every hat $\Lambda(q_1, q, q_2)$ there exists a unique anti-hat $\mathrm{V}(p_1, p, p_2)$ with $p = q_1 \wedge q_2$ such that $[p, q]$ is a 2-facet; and

(H4) there exists a 2-facet labeling λ of $\mathcal{P}$ and a 2-facet rank function of $\mathcal{P}$ with respect to λ.

The importance of the class of $\mathcal{HH}$-lattice comes from the following result.

Theorem 3.5.5 ([37, Corollary 1])

Every $\mathcal{HH}$-lattice is a bounded-homomorphic image of a free lattice.

We prove Theorem 3.5.1 by showing conditions (H1)–(H4) step by step. We begin with an observation on the 2-facets of $\mathcal{W}$, which is [37, Proposition 6] in the finite case.

Proposition 3.5.6

Let W be a Coxeter group, and let $\mathrm{V}(u_1,u,u_2)$ be an anti-hat of $\mathcal{W}$ such that u_1 and u_2 have an upper bound. Then, the interval $[u,u_1\vee_S u_2]$ is a 2-facet.

Proof. Since $\mathrm{V}(u_1,u,u_2)$ is an anti-hat it follows by definition that $u\lessdot_S u_1,u_2$, which implies that there are two simple reflections $s_1,s_2\in S$ with $u_1=us_1$ and $u_2=us_2$. Proposition 1.2.15 implies $[u,u_1\vee_S u_2]\cong[\varepsilon,s_1\vee_S s_2]$, which is the weak order lattice of a dihedral group and clearly a 2-facet. □

Now let us generalize Proposition 3.5.6 to $\mathcal{C}_\gamma$.

Proposition 3.5.7

Let $\gamma=s_1s_2\cdots s_n\in W$ be a Coxeter element, and let $w\in C_\gamma$. If $\mathrm{V}(u_1,u,u_2)$ is an anti-hat in $[\varepsilon,w]_\gamma$, then the interval $[u,u_1\vee_\gamma u_2]_\gamma$ is a 2-facet.

Proof. Let $v=u_1\vee_\gamma u_2$, which exists by Theorem 3.3.1 since $u_1,u_2\leq_\gamma w$. Again we proceed by induction on rank and length. If W is a Coxeter group of rank 2 generated by s_1 and s_2, then we can write $W=I_2(k)$ for some k, where $(s_1s_2)^k=\varepsilon$. There exists an anti-hat in $[\varepsilon,w]_\gamma$ only if $k<\infty$, and w must then necessarily be the longest element of W. (Otherwise, $[\varepsilon,w]_\gamma$ is a chain.) Hence this anti-hat is $\mathrm{V}(s_1,\varepsilon,s_2)$, which implies $v=w$, and the result follows. If $\ell_S(w)=2$, then $[\varepsilon,w]$ is a diamond, and the result follows. Hence let W have rank n, and let $\ell_S(w)=k$. Suppose that the claim is true for all parabolic subgroups of W of rank $<n$, and for all γ'-sortable elements $w'\in W$ for some Coxeter element $\gamma'\in W$ with $\ell_S(w')<k$. Without loss of generality we can assume $u\neq\varepsilon$ since otherwise $u_1,u_2\in S$, and the result follows immediately. We distinguish two cases:

(i) Let $s_1\leq_\gamma w$. If $s_1\leq_\gamma u$, then Lemma 3.3.8 implies $[u,v]_\gamma\cong[s_1u,s_1v]_{s_1\gamma s_1}$, and the result follows by induction on length.

Now suppose that $s_1\not\leq_\gamma u$ and $s_1\leq_\gamma v$. Suppose further that $s_1\not\leq_\gamma u_1,u_2$. By Lemma 3.3.7, we have $u_1\lessdot_\gamma s_1\vee_\gamma u_1=u_1'$ and $u_2\lessdot_\gamma s_1\vee_\gamma u_2=u_2'$. If $u_1'\neq u_2'$, then we obtain a contradiction to $s_1\not\leq_\gamma u$ since u_1' and u_2' are both upper bounds for u and s_1 and $[\varepsilon,w]_\gamma$ is a lattice. If $u_1'=u_2'$, then $u_1'=u_1\vee_\gamma u_2=v$, and $[u,v]_\gamma$ is a 2-facet. However, the set $\{\varepsilon,s_1,u,u_1,u_2,v\}$ induces a sublattice of $[\varepsilon,w]_\gamma$ isomorphic to $\mathcal{L}_3$ depicted in Figure 3(c), contradicting the semidistributivity of $\mathcal{C}_\gamma$. Without loss of generality let $s_1\leq_\gamma u_1$ and $s_1\not\leq_\gamma u_2$. Lemma 3.3.7 implies $u_1=s_1\vee_\gamma u$, and $u_2\lessdot_\gamma s_1\vee_\gamma u_2=v$. (Indeed, suppose that $s_1\vee_\gamma u<_\gamma v$, then s_1 and u are both lower bounds for u_1 and $s_1\vee_\gamma u_2$, which contradicts $s_1\not\leq_\gamma u$.) Thus $[u,v]_\gamma$ is a 2-facet. If $s_1\leq_\gamma u_1,u_2$, then we obtain a contradiction to $s_1\not\leq_\gamma u$.

Finally let $s_1\not\leq_\gamma v$. Lemma 3.3.8 implies that $[u,v]_\gamma\cong\left[u_{\langle s_1\rangle},v_{\langle s_1\rangle}\right]_{s_1\gamma}$, and the result follows by induction on rank.

(ii) Let $s_1\not\leq_\gamma w$. Lemma 3.3.8 implies that $[\varepsilon,w]_\gamma\cong\left[\varepsilon,w_{\langle s_1\rangle}\right]_{s_1\gamma}$, and the result follows by induction on rank. □

Proposition 3.5.8

Let $\gamma = s_1 s_2 \cdots s_n \in W$ be a Coxeter element, and let $w \in C_\gamma$. If $\Lambda(v_1, v, v_2)$ is a hat in $[\varepsilon, w]_\gamma$, then the interval $[v_1 \wedge_\gamma v_2, v]_\gamma$ is a 2-facet.

Proof. Let $u = v_1 \wedge_\gamma v_2$, and we proceed by induction on rank and length. If W has rank 2, or $\ell_S(w) = 2$, then the result is trivially true, see the first lines of the proof of Proposition 3.5.7. Hence let W have rank n and let $\ell_S(w) = k$. Suppose that the claim is true for all parabolic subgroups of W of rank $< n$, and for all γ'-sortable elements $w' \in W$ for some Coxeter element $\gamma' \in W$ with $\ell_S(w') < k$. We distinguish two cases:

(i) Let $s_1 \leq_\gamma w$. If $s_1 \leq_\gamma u$, then Lemma 3.3.8 implies $[u, v]_\gamma \cong [s_1 u, s_1 v]_{s_1 \gamma s_1}$, and the result follows by induction on length.

Now suppose that $s_1 \not\leq_\gamma u$ and $s_1 \leq_\gamma v$. We can assume without loss of generality that $s_1 \leq_\gamma v_1$ and $s_1 \not\leq_\gamma v_2$. Let $u_1 = s_1 \vee_\gamma u$. It follows from Lemma 3.3.7 that $u \lessdot_\gamma u_1$. If u_1 is the only upper cover of u in $[u, v]_\gamma$, then we obtain a contradiction to $u = v_1 \wedge_\gamma v_2$. So let u_2 be another upper cover of u in $[u, v]_\gamma$. We can find u_2 such that $u_2 \leq_\gamma v_2$, and we necessarily have $s_1 \not\leq_\gamma u_2$. Let $u_2' = s_1 \vee_\gamma u_2$. It follows from Lemma 3.3.7 that $u_2 \lessdot_\gamma u_2' \leq_\gamma v$. Now u_1 and u_2' are both upper bounds of s_1 and u, and it follows that $u_1 \leq_\gamma u_2'$. Assume that $u_2' <_\gamma v$. If $u_2' \leq_\gamma v_1$, then we obtain a contradiction to $v_1 \wedge_\gamma v_2 = u$, and likewise if $u_2' \leq_\gamma v_2$. Thus there must be some element $v' \in C_\gamma$ with $u_2' \leq_\gamma v' \lessdot_\gamma v$ and $v' \neq v_1, v_2$. Then, however, the set $\{u, u_1, u_2, u_2', v_1, v_2, v', v\}$ induces a sublattice of $[\varepsilon, w]_\gamma$ isomorphic to $\mathcal{L}_4$ depicted in Figure 3(d), contradicting the semidistributivity of $\mathcal{C}_\gamma$. Hence it follows that $u_2' = v$ and $u_2 = v_2$, which implies that $[u, v]$ is a 2-facet.

Finally let $s_1 \not\leq_\gamma v$. Lemma 3.3.8 implies that $[u, v]_\gamma \cong \big[u_{\langle s_1 \rangle}, v_{\langle s_1 \rangle}\big]_{s_1 \gamma}$, and the result follows by induction on rank.

(ii) Let $s_1 \not\leq_\gamma w$. Lemma 3.3.8 implies that $[\varepsilon, w]_\gamma \cong \big[\varepsilon, w_{\langle s_1 \rangle}\big]_{s_1 \gamma}$, and the result follows by induction on rank. ☐

In [37], the following edge-labeling of $\mathcal{W}$ was considered:

$$r : \mathcal{E}(\mathcal{W}) \to T, (u, v) \mapsto t, \tag{3.9}$$

where t is the unique cover reflection of v satisfying $u = tv$. The following result was proven in [37] for finite Coxeter groups, but its proof can be generalized straightforwardly to closed intervals of $\mathcal{W}$ in the infinite case.

Proposition 3.5.9 ([37, Corollary 3 and Theorem 5])

The labeling r from (3.9) is a 2-facet labeling of $\mathcal{W}$. Moreover, the length function ℓ_S is a 2-facet rank function of $\mathcal{W}$ with respect to r.

For our proof of Theorem 3.5.1 we use a similar labeling, but first we need some preparation.

Lemma 3.5.10

Let $\gamma = s_1 s_2 \cdots s_n$, and let $u, v \in C_\gamma$ with $u \lessdot_\gamma v$. There exists a unique element $w_{u,v} \in W$ with $u \leq_S w_{u,v} \lessdot_S v$ and $\pi_\downarrow^\gamma(w_{u,v}) = u$.

Proof. If $\ell_S(v) = \ell_S(u) + 1$, then $u \lessdot_S v$, and the result follows by setting $w_{u,v} = u$. So suppose that $\ell_S(v) > \ell_S(u) + 1$. Since $\mathcal{W}$ is graded it follows that there exists some $w \in W$

with $u <_S w \lessdot_S v$ and $w \notin C_\gamma$. It follows from Theorem 3.3.2 that $u = \pi_\downarrow^\gamma(u) \leq_\gamma \pi_\downarrow^\gamma(w) \leq_\gamma \pi_\downarrow^\gamma(v) = v$, and since $u \lessdot_\gamma v$ we conclude $\pi_\downarrow^\gamma(w) = u$. Suppose that there is another element w' with $u <_S w' \lessdot_S v$ and $\pi_\downarrow^\gamma(w') = u$. Then we have $w \vee_S w' = v$, and Theorem 3.3.3 implies

$$u = u \vee_\gamma u = \pi_\downarrow^\gamma(w) \vee_\gamma \pi_\downarrow^\gamma(w') = \pi_\downarrow^\gamma(w \vee_S w') = \pi_\downarrow^\gamma(v) = v,$$

which contradicts $u \lessdot_\gamma v$. Thus we have $w_{u,v} = w$, and we are done. □

The element $w_{u,v}$ from the previous lemma is the unique maximal element in the fiber of u with respect to $\pi_\downarrow^\gamma$. By definition there exists some $t_{u,v} \in T$, and some $s \in S$ with $w_{u,v} = vs = t_{u,v}v$ and thus $t_{u,v} \in \operatorname{cov}(v)$. Define an edge-labeling of $\mathcal{C}_\gamma$ by

$$\varphi_\gamma : \mathcal{E}(\mathcal{C}_\gamma) \to T, \quad (u,v) \mapsto t_{u,v}. \tag{3.10}$$

By definition we have $\varphi_\gamma(u,v) = t_{u,v} = r(w_{u,v}, v)$. Let us investigate how this labeling behaves with respect to induction on rank and length.

Lemma 3.5.11

Let $\gamma = s_1 s_2 \cdots s_n \in W$, and let $u, v \in C_\gamma$ with $u \lessdot_\gamma v$. We have

$$\varphi_\gamma(u,v) = \begin{cases} s_1 \varphi_{s_1\gamma s_1}(s_1 u, s_1 v) s_1, & \text{if } s_1 \leq_\gamma u,\\ s_1, & \text{if } s_1 \not\leq_\gamma u \text{ and } s_1 \leq_\gamma v,\\ \varphi_{s_1\gamma}\big(u_{\langle s_1\rangle}, v_{\langle s_1\rangle}\big), & \text{if } s_1 \not\leq_\gamma v.\end{cases}$$

Moreover, let $t = \varphi(u,v), t' = \varphi_{s_1\gamma s_1}(s_1 u, s_1 v)$ and $t_{\langle s_1\rangle} = \varphi_{s_1\gamma}\big(u_{\langle s_1\rangle}, v_{\langle s_1\rangle}\big)$. Then, we have

$$\ell_S(t) = \begin{cases} \ell_S(s_1 t' s_1), & \text{if } s_1 \leq_\gamma u,\\ 1, & \text{if } s_1 \not\leq_\gamma u \text{ and } s_1 \leq_\gamma v,\\ \ell_S\big(t_{\langle s_1\rangle}\big), & \text{if } s_1 \not\leq_\gamma v.\end{cases}$$

Proof. Let $w_{u,v} \in W$ be the lower cover of v in $\mathcal{W}$ from Lemma 3.5.10, and let $t_{u,v} = \varphi_\gamma(u,v)$.

Let $s_1 \leq_\gamma u$, and write $u' = s_1 u, v' = s_1 v$ and $w_{u',v'} = s_1 w_{u,v}$. In view of Proposition 1.2.20 we conclude that $w_{u',v'} \lessdot_S v'$. Let $t_{u',v'} = r(w_{u',v'}, v')$. This means that there exists some $s \in S$ with $w_{u',v'} = v's = t_{u',v'}v'$, which implies that $s_1 vs = t_{u',v'}s_1 v$. Hence $vs = s_1 t_{u',v'} s_1 v$ and $s_1 t_{u',v'} s_1 = t_{u,v}$, and we conclude

$$\varphi_\gamma(u,v) = r(w_{u,v}, v) = t_{u,v} = s_1 t_{u',v'} s_1 = s_1 r(w_{u',v'}, v') s_1 = s_1 \varphi_{s_1\gamma s_1}(u', v') s_1.$$

(Note that Lemma 3.3.8 implies $u' \lessdot_{s_1\gamma s_1} v'$.) In particular, we have $t_{u,v} = vsv^{-1} = s_1 t_{u',v'} s_1$, which implies $\ell_S(t_{u,v}) = \ell_S(s_1 t_{u',v'} s_1)$.

Now let $s_1 \not\leq_\gamma u$, and suppose that $s_1 \leq_\gamma v$. Lemma 3.3.7 implies $v = s_1 \vee_\gamma u$, which implies using Theorem 3.3.1 that $v = s_1 \vee_S u$. Hence $s_1 \not\leq_S w_{u,v}$. Further Proposition 1.2.20 implies $v = s_1 w_{u,v}$, and hence $t_{u,v} = \varphi_\gamma(u,v) = r(w_{u,v}, v) = s_1$. Now the relation for $\ell_S(t_{u,v})$ is immediate. Suppose that $s_1 \not\leq_\gamma v$. It follows from Proposition 3.2.6 that $u = u_{\langle s_1\rangle} \in W_{\langle s_1\rangle}$ and $v = v_{\langle s_1\rangle} \in W_{\langle s_1\rangle}$. In particular $t_{u,v} \in W_{\langle s_1\rangle}$. Hence

$$\varphi_\gamma(u,v) = \varphi_\gamma\big(u_{\langle s_1\rangle}, v_{\langle s_1\rangle}\big) = \varphi_{s_1\gamma}\big(u_{\langle s_1\rangle}, v_{\langle s_1\rangle}\big),$$

and the result for $\ell_S(t_{u,v})$ is immediate. □

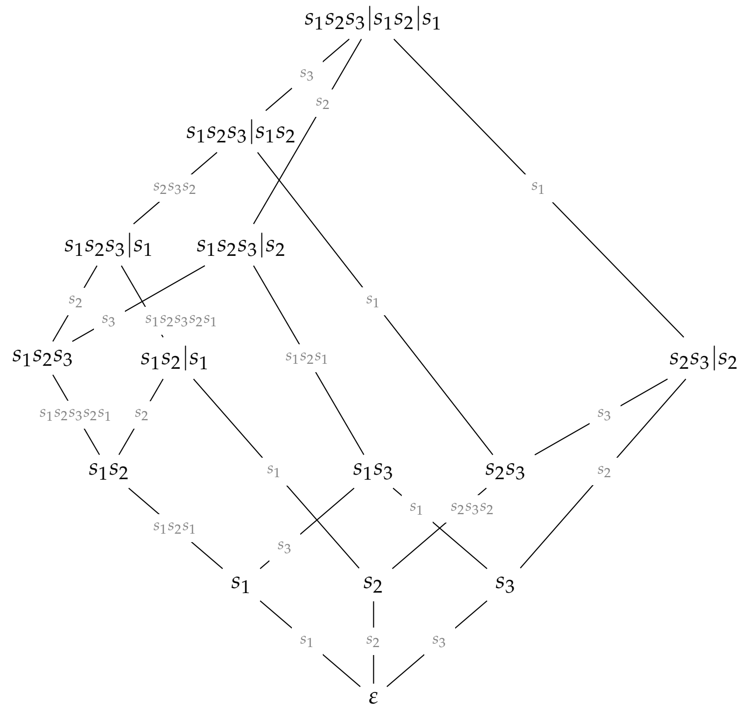


FIGURE 36. An A_3-Cambrian lattice with the 2-facet labeling defined in (3.10).

EXAMPLE 3.5.12

Figures 36 and 37 show the Tamari lattice and an interval of a Cambrian semilattice associated with the infinite Coxeter group $\tilde{C}_3$, respectively. Both lattices are labeled by φ_γ.

PROPOSITION 3.5.13

Let $\gamma = s_1 s_2 \cdots s_n$. The map φ_γ defined in (3.10) *is a 2-facet labeling of $\mathcal{C}_\gamma$.*

PROOF. Let $[u,v]_\gamma$ be 2-facet of $\mathcal{C}_\gamma$. By definition there exists a hat $\Lambda(v_1,v,v_2)$ and an anti-hat $\mathrm{V}(u_1,u,u_2)$ with $v = u_1 \vee_\gamma u_2$ and $u = v_1 \wedge_\gamma v_2$. In view of the proofs of Propositions 3.5.7 and 3.5.8 we can assume that $u_2 = v_2$. We need to show that $\varphi_\gamma(u,u_1) = \varphi_\gamma(v_2,v)$ and $\varphi_\gamma(u,u_2) = \varphi_\gamma(v_1,v)$.

In view of Lemma 3.5.10 there exist elements $w_1, w_2 \in W$ with $w_1, w_2 \lessdot_S v$ and $\pi_\downarrow^\gamma(w_1) = u_1$ and $\pi_\downarrow^\gamma(w_2) = u_2$. If we set $w = w_1 \wedge_S w_2$, then we notice immediately that $[w,v]_S$ is a 2-facet in $\mathcal{W}$. If $w = u$, then we have $w_1 = v_1 = u_1$ and $w_2 = v_2 = u_2$, and the result follows from Proposition 3.5.9. So suppose that $u <_S w$. It follows from Theorem 3.3.3 that $\pi_\downarrow^\gamma(w) = u$. Let $z \in W$ be the unique element satisfying $w \lessdot_S z \leq_S w_1$. Theorem 3.3.2 implies $u \leq_\gamma \pi_\downarrow^\gamma(z) \leq_\gamma v_1$. If $\pi_\downarrow^\gamma(z) = u$, then we obtain

$$v = \pi_\downarrow^\gamma(v) = \pi_\downarrow^\gamma(z \vee_S w_2) = \pi_\downarrow^\gamma(z) \vee_\gamma \pi_\downarrow^\gamma(w_2) = u \vee_\gamma v_2 = v_2,$$

which is a contradiction. Hence $u <_\gamma z$. If $u = z \wedge_S v_1$, then it follows from Proposition 3.5.9 that $\varphi_\gamma(v_2,v) = \varphi_\gamma(u,u_1)$. If $u <_S z \wedge_S v_1$, then we can consider the interval $[u,z]_S$ and repeat

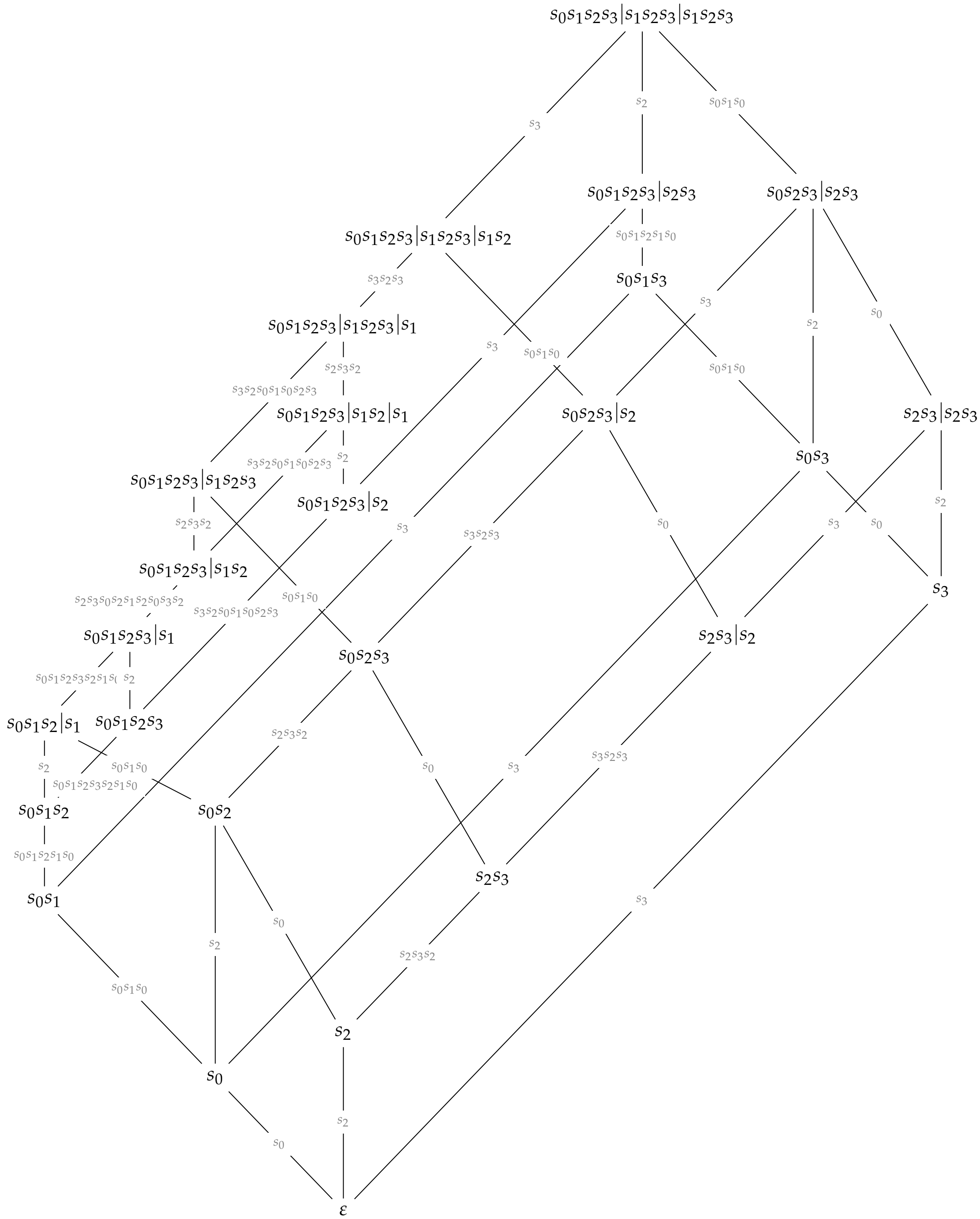


Figure 37. An interval of a $\tilde{C}_3$-Cambrian semilattice with the labeling defined in (3.10).

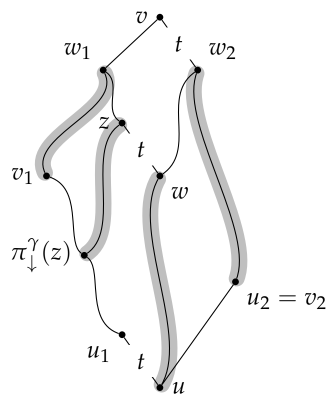


FIGURE 38. Illustrating the proof of Proposition 3.5.13. The rounded edges indicate chains, the straight edges indicate cover relations, and the shaded edges indicate fibers of $\pi_\downarrow^\gamma$.

the previous reasoning. (Notice that $r(w,z) = r(w_2,v)$ by Proposition 3.5.9, and notice that there cannot be an element $w' \in W$ with $u_1 <_S w' <_S w$, because then Theorem 3.3.2 would imply
$$u \lessdot_\gamma u_1 = \pi_\downarrow^\gamma(u_1) \leq_\gamma \pi_\downarrow^\gamma(w') \leq_\gamma \pi_\downarrow^\gamma(w) = u,$$
which is a contradiction.) See Figure 38 for an illustration. The reasoning that $\varphi_\gamma(u,u_2) = \varphi_\gamma(v_1,v)$ is analogous. $\square$

PROPOSITION 3.5.14

Let $\gamma = s_1s_2\cdots s_n$. The length function ℓ_S is a 2-facet rank function of $\mathcal{C}_\gamma$ with respect to φ_γ.

PROOF. Let $u,v \in C_\gamma$ with $u \leq_\gamma v$ such that $[u,v]_\gamma$ is a 2-facet. By definition there exists a hat $\Lambda(v_1,v,v_2)$ and an anti-hat $\mathrm{V}(u_1,u,u_2)$ with $v = u_1 \vee_\gamma u_2$ and $u = v_1 \wedge_\gamma v_2$. In view of the proofs of Propositions 3.5.7 and 3.5.8 we can assume that $u_2 = v_2$. We have to show that ℓ_S satisfies the conditions in Definition 3.5.2, and we notice that nothing has to be checked for the chain $u \lessdot_\gamma u_2 \lessdot_\gamma v$.

We proceed by induction on rank and length, and if W has rank 2 or $\ell_S(v) = 2$, then there is nothing to show. Hence let W have rank n and let $\ell_S(w) = k$. Suppose that the claim is true for all parabolic subgroups of W of rank $< n$, and for all 2-facets $[u',v']_{\gamma'}$ for some Coxeter element $\gamma' \in W$ with $\ell_S(v') < k$. We distinguish three cases:

(i) Let $s_1 \leq_\gamma u$. Denote the other chain in $[u,v]_\gamma$ by $C : u \lessdot_\gamma u_1 \lessdot_\gamma \cdots \lessdot_\gamma v_1 \lessdot_\gamma v$. It follows from Lemma 3.3.8 that $[u,v]_\gamma \cong [s_1u, s_1v]_{s_1\gamma s_1}$, and the corresponding chain C' : $s_1u \lessdot_{s_1\gamma s_1} s_1u_1 \lessdot_{s_1\gamma s_1} \cdots \lessdot_{s_1\gamma s_1} s_1v_1 \lessdot_{s_1\gamma s_1} s_1v$ has the label sequence $\varphi_{s_1\gamma s_1}(C') = (t'_1, t'_2, \ldots, t'_k)$. Lemma 3.5.11 implies now that $\varphi_\gamma(C) = (s_1t'_1s_1, s_1t'_2s_1, \ldots, s_1t'_ks_1)$, and by induction and Lemma 3.5.11 follows that $\ell_S(s_1t'_ks_1) = \ell_S(t'_k) < \ell_S(t'_2) = \ell_S(s_1t'_2s_1)$, and likewise for the other relations.

(ii) Let $s_1 \not\leq_\gamma u$ and $s_1 \leq_\gamma v$. Without loss of generality we can assume that $s_1 \leq_\gamma u_1$ and $s_1 \not\leq_\gamma u_2$. Lemma 3.3.7 implies that $u_1 = s_1 \vee_\gamma u$ and $u' = s_1 \vee_\gamma u_2$, and in particular that $u_2 \lessdot_\gamma u'$. Then, however, it follows that u_1 and u' are both upper bounds of s_1 and u, and since $[u,v]_\gamma$ is a lattice it follows that $u_1 <_\gamma u'$. Then $v = u_1 \vee_\gamma u_2 \leq_\gamma u'$ implies that $v = u'$. Hence the two chains in $[u,v]_\gamma$ have length 2, and nothing has to be checked.

(iii) Let $s_1 \not\leq_\gamma v$. Then $s_1 \not\leq_\gamma u$, and Lemma 3.3.8 implies that $[u,v]_\gamma \cong [u_{\langle s_1\rangle}, v_{\langle s_1\rangle}]_{s_1\gamma}$, and the result follows by induction on rank. $\square$

Now everything is set to prove Theorem 3.5.1.

PROOF OF THEOREM 3.5.1. We need to check that every closed interval of $\mathcal{C}_\gamma$ satisfies conditions (H1)–(H4) from Definition 3.5.4. Let $u,v \in C_\gamma$ with $u \leq_\gamma v$, and consider the interval $[u,v]_\gamma$. Proposition 1.2.18 implies that $[u,v]_\gamma$ is finite, and Proposition 3.3.6 implies that $[u,v]_\gamma$ is semidistributive. Hence (H1) is satisfied. Propositions 3.5.7 and 3.5.8 imply that $[u,v]_\gamma$ satisfies (H2) and (H3). Finally Propositions 3.5.13 and 3.5.14 imply that $[u,v]_\gamma$ satisfies (H4). $\square$

REMARK 3.5.15

The previous reasoning implies that a result analogous to Theorem 3.5.1 holds also for closed intervals of the weak order semilattice of W. Thus we can generalize [37, Theorem 6] to the infinite case.

REMARK 3.5.16

It follows from Theorems 1.1.28 and 3.5.1 that every closed interval of a γ-Cambrian semilattice can be constructed from the one-element lattice **1** by successive interval doubling. The edge-labeling φ_γ defined in (3.10) indicates in which order this interval doubling takes place, by reversing this procedure. We start with an interval $[u,v]_\gamma$, and successively contract intervals, starting with edges whose labels have maximum length. (An interval contraction is the inverse of an interval doubling.) See Appendix B for an illustration.

3.5.2. Breadth. We complete our study of the γ-Cambrian semilattices by determining their breadth. Recall that for a finite lattice $\mathcal{P} = (P,\leq)$ the *breadth of $\mathcal{P}$* is the least number $b(\mathcal{P})$ such that the following is satisfied: if $p \in P$ can be written as $p = p_1 \vee p_2 \vee \cdots \vee p_k$ for $k > b(\mathcal{P})$, then p can already be written as the join of a $b(\mathcal{P})$-element subset of $\{p_1, p_2, \ldots, p_k\}$. In other words $b(\mathcal{P})$ is the least number such that every element in $\mathcal{P}$ can be written as the join of at most $b(\mathcal{P})$ elements. In this section we prove the following theorem.

THEOREM 3.5.17

Let (W,S) be a Coxeter system of rank n, where $S = \{s_1, s_2, \ldots, s_n\}$, and let $\gamma = s_1 s_2 \cdots s_n \in W$ be a Coxeter element. If $u,v \in W$ with $u \leq_S v$, then

$$b\big([u,v]_S\big) = \max\big\{|\mathrm{cov}(u^{-1}w)| \mid u \leq_S w \leq_S v\big\}.$$

If moreover $u,v \in C_\gamma$, then

$$b\big([u,v]_\gamma\big) = \max\big\{|\mathrm{cov}(w) \setminus \mathrm{inv}(u)| \mid u \leq_\gamma w \leq_\gamma v\big\}.$$

We prove Theorem 3.5.17 by exploiting the fact that every element in $\mathcal{W}$ has a canonical join-representation, see Section 1.1.6 for the definition. For that we use the following connection between canonical join-representations and the breadth of a lattice.

PROPOSITION 3.5.18

Let $\mathcal{P} = (P,\leq)$ be a lattice such that every $p \in P$ has a canonical join-representation, denoted by Z_p. Then, $b(\mathcal{P}) = \max\big\{|Z_p| \mid p \in P\big\}$.

PROOF. Let $p \in P$ such that $k = |Z_p|$ is maximal. If $b(\mathcal{P}) < k$, then by definition of the breadth, we can write p as a join of a $b(\mathcal{P})$-element subset of Z_p, which contradicts the assumption that Z_p is the canonical join-representation of p.

Now suppose that $b(\mathcal{P}) > k$. We can find $x \in P$ and $X \subseteq P$ with $x = \bigvee X$ and $|X| = b(\mathcal{P})$. Then it follows from the maximality of k that X cannot be the canonical join-representation of x, and hence $Z_x \subsetneq X$. By definition of the breadth follows $x = \bigvee Z_x < \bigvee X = x$, which is a contradiction.

Hence we have $b(\mathcal{P}) = k$. □

The next proposition shows that intervals of lattices with canonical join-representations have canonical join-representations again.

PROPOSITION 3.5.19

Let $\mathcal{P} = (P, \leq)$ be a lattice such that every $p \in P$ has a canonical join-representation. Let $[p,q]$ be an interval in $\mathcal{P}$, let $z \in P$ with $p \leq z \leq q$, and suppose that $Z_z = \{j_1, j_2, \ldots, j_k\}$ is the canonical join-representation of z in $\mathcal{P}$. Define $k_i = p \vee j_i$ for $i \in \{1,2,\ldots,k\}$. The set $\{k_i \mid i \in \{1,2,\ldots,k\}$ and $k_i \neq p\}$ is the canonical join-representation of z in $[p,q]$.

PROOF. Define $Z'_z = \{k_i \mid i \in \{1,2,\ldots,k\} \text{ and } k_i \neq p\} = \{k_{i_1}, k_{i_2}, \ldots, k_{i_l}\}$.

It is straightforward to verify that Z'_z is indeed a join-representation of z in $[p,q]$. Suppose that there is some other join-representation X of z in $[p,q]$. This implies in particular that X is a join-representation of z in $\mathcal{P}$, and it follows that Z_z refines X. Let $j_i \in Z_z$ such that $j_i \not\leq p$. There exists some $x \in X$ with $j_i \leq x$, and $k_i = p \vee j_i \in Z'_z$. Since $p \leq x$ we conclude that x is an upper bound for both j_i and p, and it follows that $k_i \leq x$. Thus Z'_z refines X. □

Now we can prove Theorem 3.5.17.

PROOF OF THEOREM 3.5.17. First let $w \in W$, and consider the interval $[\varepsilon, w]_S$. If Z_w denotes the canonical join-representation of w, then it follows from Theorem 3.3.4 that $|Z_w| = |\mathrm{cov}(w)|$, and the claim follows immediately from Proposition 3.5.18.

Let $[u,v]_S$ be an interval of $\mathcal{W}$. Proposition 1.2.15 implies $[u,v]_S \cong [\varepsilon, u^{-1}v]_S$. It follows from the reasoning in the first part of this proof that

$$b\big([\varepsilon, u^{-1}v]_S\big) = \max\big\{|\mathrm{cov}(w)| \mid \varepsilon \leq_S w \leq_S u^{-1}v\big\},$$

and in view of the present isomorphism $b\big([u,v]_S\big)$ takes the same value. If w lies in $[\varepsilon, u^{-1}v]_S$, then it can be written as $w = u^{-1}w'$ for some w' in $[u,v]_S$.

Now suppose that $u, v \in C_\gamma$. For $w \in C_\gamma$ with $u \leq_\gamma w \leq_\gamma v$ let Z_w denote the canonical join-representation of w in $\mathcal{C}_\gamma$, and let Z'_w denote the canonical join-representation of w in $[u,v]_\gamma$. It follows from Theorem 3.3.4 and Proposition 3.3.5 that $|Z_w| = |\mathrm{cov}(w)|$, and Proposition 3.5.19 implies that every $j \in Z_w$ with $j \not\leq_\gamma u$ contributes to Z'_w. It follows from the reasoning in [97, Section 8] that $j \in Z_w$ is the unique join-irreducible element below w having $\mathrm{cov}(j) = t$ for some $t \in \mathrm{cov}(w)$. Thus if $j \leq_\gamma u$, then $t \in \mathrm{cov}(w) \cap \mathrm{inv}(u)$, and if $j \not\leq_\gamma u$, then $t \in \mathrm{inv}(v) \setminus \mathrm{cov}(w)$, which implies the claim. □

CHAPTER 4

The Lattices of Noncrossing Partitions

4.1. Introduction

The study of noncrossing partitions was initiated by Kreweras' article [70], where he thoroughly investigated *noncrossing set partitions*, i.e. set partitions of $\{1,2,\ldots,n\}$ with the property that for any four elements $i<j<k<l$ the elements i and k do not lie together in one block while j and l lie together in another block.

Example 4.1.1

Let $n=10$, and consider the set partition

$$P=\big\{\{1,3,5,7\},\{2,8\},\{4,6,9,11,12\},\{10\}\big\}.$$

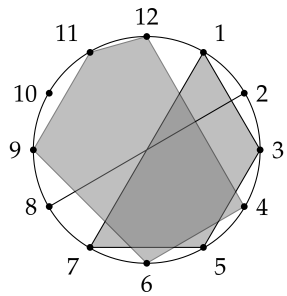
This set partition is crossing, since for instance $i=1, j=2, k=3$ and $l=8$ have the property that i and k lie together in one block while j and l lie together in another block. In contrast the set partition

$$P'=\big\{\{1,8\},\{2,3,5,6,7\},\{4\},\{9,11,12\},\{10\}\big\},$$

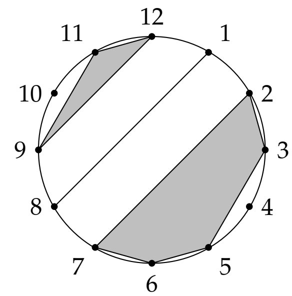
is noncrossing.

Among other things Kreweras considered these noncrossing set partitions as a poset where the corresponding partial order is refinement, i.e. two noncrossing set partitions P_1 and P_2 satisfy $P_1\leq_{\mathrm{ref}}P_2$ if each block of P_1 is contained in some block of P_2. We denote the set of noncrossing set partitions of $\{1,2,\ldots,n\}$ by NC_n, and we write $\mathcal{NC}_n=\big(NC_n,\leq_{\mathrm{ref}}\big)$. Kreweras proved that the cardinality of NC_n is the n-th Catalan number $\mathrm{Cat}(n)$, see [70, Corollaire 4.2], he showed that these posets are graded lattices of length n, and he computed the values of the Möbius function for these posets. Later Simion and Ullmann proved that $\mathcal{NC}_n$ is self-dual, see [108].

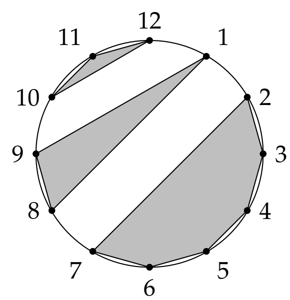
We can graphically represent set partitions in the following way: we draw the numbers from 1 to n on a circle, we sort the elements of each block increasingly, and we connect two (cyclically) neighboring entries in a block by a diagonal. (Here, "cyclically" means that we also connect the largest element in a block with the smallest element in the same block.) Finally we draw the convex hulls of the diagonals between elements lying in the same block. In this representation, a set partition is noncrossing if and only if these convex hulls do not cross, see Figure 39 for an illustration, and see Figure 40 for the lattice $\mathcal{NC}_4$. Moreover, we can



(a) The crossing set partition P from Example 4.1.1.



(b) The noncrossing set partition P' from Example 4.1.1.



(c) The 3-divisible set partition P'' from Example 4.1.2.

FIGURE 39. Cycle diagrams of set partitions.

interpret the blocks of a set partition as cycles of a permutation in A_{n-1}, and it was observed by BIANE that the poset $\mathcal{NC}_n$ can be interpreted as an interval of the absolute order on A_{n-1}, see [17, Theorem 1]. Compare also Figure 40 with the highlighted interval in the poset shown in Figure 8 on page 26.

At about the same time as BIANE's observation that $\mathcal{NC}_n$ can be interpreted as a poset associated with the Coxeter group A_{n-1}, REINER independently defined noncrossing partitions of type B and D in an analogous fashion as before, i.e. by defining cycle diagrams of type B and D, and by saying when a configuration of diagonals in such a cycle diagram is noncrossing, see [98]. He interpreted these "noncrossing partitions" as elements in the intersection lattice of the reflection arrangement of the corresponding Coxeter group. This was motivated by the observation that the lattice of all set partitions under refinement is isomorphic to the intersection lattice of the type-A reflection arrangement. Some time later BRADY and WATT discovered that for some Coxeter group W and some Coxeter element $\gamma \in W$ the interval $[\varepsilon, \gamma]$ in the absolute order is a lattice, see [32, 33]. In particular if $W = A_{n-1}$, then this lattice agrees with $\mathcal{NC}_n$, and if $W = B_n$, then this lattice agrees with REINER's type-B noncrossing partition lattice. However, for $W = D_n$, the lattices of REINER do not coincide with the ones of BRADY and WATT. A realization of the noncrossing partition lattices for D_n in terms of cycle diagrams was given in [7]. BESSIS has independently made more or less the same observation, however from a different point of view. In fact, he considered a slightly more general setting involving the so-called well-generated complex reflection groups, and

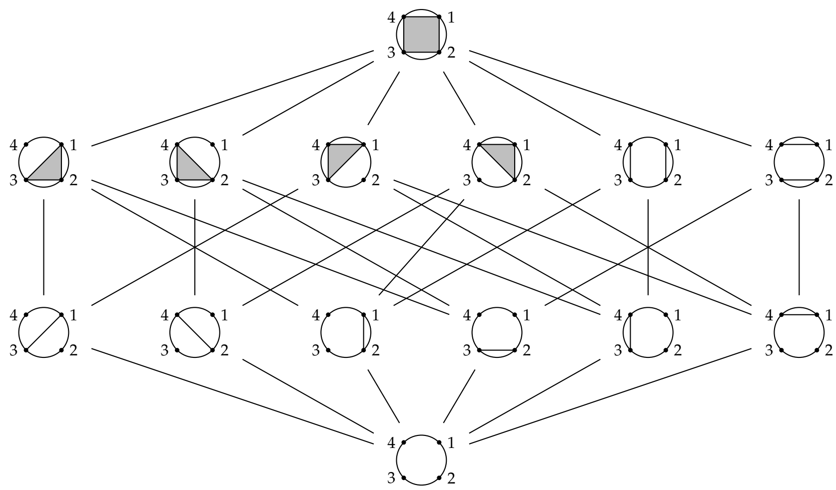


FIGURE 40. The lattice $\mathcal{NC}_4$.

he proved that the cardinality of these lattices is given by the generalized Catalan number associated with the corresponding reflection group, see [12, Proposition 5.2.1]. Interestingly the lattices of generalized noncrossing partitions are not only interesting from a lattice-theoretic point of view because they possess many beautiful structural properties, but they also appear in different, seemingly unrelated fields of mathematics, such as group theory [19], topology [12, 15, 32], free probability [18], representation theory of quivers [64], or cluster algebras [94]. See [77, 106] for surveys on this matter, or see [1, Section 4.1] for a historical overview on the theory of noncrossing partitions.

A second, different generalization of the noncrossing set partitions was presented by EDELMAN in [47], where he introduced the *m-divisible noncrossing partitions*, namely noncrossing set partitions of $\{1,2,\ldots,mn\}$ with block sizes divisible by m. Analogously to the case of ordinary noncrossing set partitions, we can illustrate these m-divisible noncrossing set partitions by means of a cycle diagram, we denote the set of all m-divisible noncrossing set partitions by $NC_n^{(m)}$, and we denote the poset of m-divisible noncrossing set partitions under refinement order by $\mathcal{NC}_n^{(m)} = \left(NC_n^{(m)}, \leq_{\mathrm{ref}}\right)$. It turns out that this poset is a graded join-semilattice, but it has in general no least element. Moreover, the cardinality of $NC_n^{(m)}$ is given by the m,n-th Fuß-Catalan number $\mathrm{Cat}^{(m)}(n)$, see [47, Lemma 4.1].

EXAMPLE 4.1.2

The set partition
$$P'' = \big\{\{1,8,9\},\{2,3,4,5,6,7\},\{10,11,12\}\big\}$$
is not only noncrossing, it is also 3-divisible, since its block sizes are $3,6$ and 3. The noncrossing set partition P' from Example 4.1.1 is only 1-divisible, since its block sizes are $2,5,1,3$ and 1. See Figure 39(c) for an illustration, and see Figure 41 for the poset $\mathcal{NC}_3^{(2)}$.

The two previously described generalizations of the lattice of noncrossing set partitions have been brought together by ARMSTRONG, who introduced in [1] the *m-divisible W-noncrossing partitions* associated with a finite Coxeter group W. BESSIS and REINER observed in [16] that this construction can be generalized straightforwardly to well-generated complex reflection

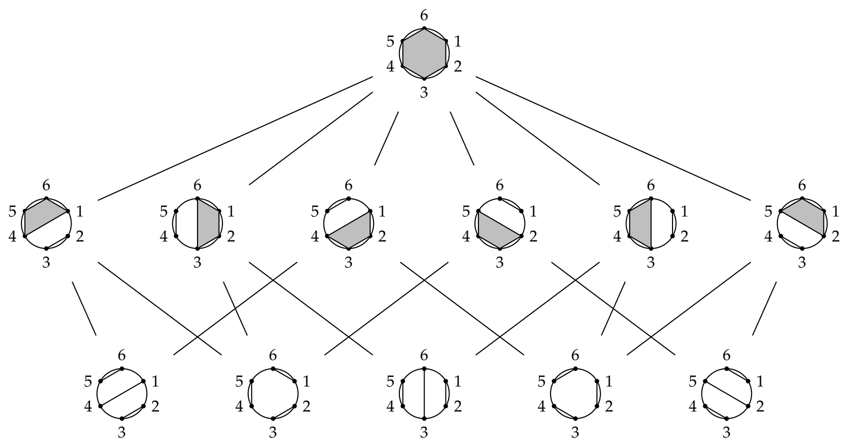


FIGURE 41. The poset $\mathcal{NC}_3^{(2)}$.

groups. The m-divisible noncrossing partitions associated with well-generated complex reflection groups are the objects of interest in this chapter.

We first formally define complex reflection groups, (and thus generalize the real reflection groups introduced in Section 1.2.3), then we define ARMSTRONG's m-divisible noncrossing partitions associated with a well-generated complex reflection group, and finally we investigate the resulting posets from a topological point of view. In particular, we prove that these lattices are EL-shellable, see Theorem 4.4.1, and we derive the value of their Möbius invariant, see Proposition 4.4.23.

4.2. Definition and Examples

In this section we formally define complex reflection groups, and we recall their classification due to SHEPHARD and TODD, see [**104**]. For any undefined notation, along with a more detailed exposition, we refer to the monograph [**73**]. Subsequently we define the central objects of this chapter: the posets of m-divisible noncrossing partitions associated with a well-generated complex reflection group, and this construction follows [**1**].

4.2.1. Complex Reflection Groups. Recall from Section 1.2.3 that so far we defined a reflection to be a linear transformation on an n-dimensional real vector space (endowed with a symmetric bilinear form $\langle\cdot,\cdot\rangle$) that sends some nonzero vector $\mathbf{v}\in V$ to its negative and that fixes the hyperplane orthogonal to $\mathbf{v}$ (with respect to $\langle\cdot,\cdot\rangle$) pointwise. In particular, we can write such a reflection as

$$s_{\mathbf{v}}(\mathbf{u}) = \mathbf{u} - 2\frac{\langle\mathbf{u},\mathbf{v}\rangle}{\langle\mathbf{v},\mathbf{v}\rangle}\mathbf{v},$$

and it is easy to check that this is an orthogonal transformation on V having order 2. It follows further that such a transformation has $n-1$ eigenvalues equal to 1, and one eigenvalue equal to -1.

We can generalize this definition by forgetting about the restriction that such a transformation has to have order 2, or equivalently, that it has to have a unique eigenvalue equal to -1. Let now V be an n-dimensional complex vector space endowed with a positive definite Hermitian form. A *complex reflection* is a unitary transformation on V that has finite order and that fixes a hyperplane in V pointwise. In other words, a unitary reflection has $n-1$

eigenvalues equal to 1 and one eigenvalue equal to some root of unity. Let $U(V)$ denote the group of unitary transformations on V. A *complex reflection group* is a finite subgroup of $U(V)$ generated by complex reflections. (In this chapter we will only consider finite groups.) Let $W \subseteq U(V)$ be a complex reflection group, and denote by V^W the set of fixed points of W, i.e. $V^W = \{\mathbf{v} \in V \mid w(\mathbf{v}) = \mathbf{v} \text{ for all } w \in W\}$. If W can written as a direct product $W \cong W_1 \times W_2$, where W_1 and W_2 are complex reflection groups acting on proper subspaces of V, then W is *reducible*. Otherwise W is *irreducible*. We explain in Disclaimer 4.2.5 that for our purposes it is sufficient to understand the irreducible complex reflection groups. Thus from now on all the groups considered are supposed to be irreducible, unless otherwise stated. The *rank of W* is defined as the dimension of the complement of V^W in V.

Shephard and Todd have characterized the finite irreducible complex reflection groups in [**104**]. Recall that a *monomial matrix* is a matrix in which each row and each column contains a unique nonzero entry. The monomial $(n \times n)$-matrices, in which the nonzero entries are d-th roots of unity and in which the product of all nonzero entries is a $\frac{d}{e}$-th root of unity, form a group, and we denote this group by $G(d,e,n)$. We have the following theorem.

Theorem 4.2.1 ([**104**, Table VII])

A group G is a finite irreducible complex reflection group if and only if G is either isomorphic to $G(d,e,n)$ for some integer e dividing d, or G is isomorphic to one of 34 exceptional groups, denoted by $G_4, G_5, \ldots, G_{37}$.

Theorem 4.2.1 extends Theorems 1.2.5 and 1.2.7 in the sense that the finite Coxeter groups are contained in this characterization. More precisely, we have the following identification, see [**73**, Example 2.11] or [**35**]:

- the group $G(1,1,n)$ for $n \geq 2$ is isomorphic to the Coxeter group A_{n-1},
- the group $G(2,1,n)$ for $n \geq 2$ is isomorphic to the Coxeter group B_n,
- the group $G(2,2,n)$ for $n \geq 4$ is isomorphic to the Coxeter group D_n,
- the group $G(d,d,2)$ for $d \geq 3$ is isomorphic to the Coxeter group $I_2(d)$,
- the group $G(2,2,3)$ is isomorphic to the Coxeter group A_3, and
- the group $G(2,2,2)$ is isomorphic to the reducible Coxeter group $A_1 \times A_1$.

The exceptional irreducible Coxeter groups are among the 34 exceptional complex reflection groups. More precisely, we have the following identification:

- the group G_{23} is isomorphic to the Coxeter group H_3,
- the group G_{28} is isomorphic to the Coxeter group F_4,
- the group G_{30} is isomorphic to the Coxeter group H_4,
- the group G_{35} is isomorphic to the Coxeter group E_6,
- the group G_{36} is isomorphic to the Coxeter group E_7, and
- the group G_{37} is isomorphic to the Coxeter group E_8.

Now let W be a irreducible complex reflection group of rank n, and let $\mathrm{Sym}(V^*)^W$ denote the invariant subring of the symmetric algebra on V^*. It is the statement of [**104**, Proposition 5.1] that $\mathrm{Sym}(V^*)^W$ is generated by polynomials. (In fact this property characterizes the finite irreducible complex reflection groups.) Now fix a homogeneous polynomial basis $f_1, f_2, \ldots, f_n$ of $\mathrm{Sym}(V^*)^W$ and denote their degrees by $d_1, d_2, \ldots, d_n$, respectively. It follows from the results in [**39**] that these degrees do not depend on the actual choice of basis. Hence we call these numbers the *degrees of W*. We can define the degrees of W in yet another way. Let $\mathrm{Sym}(V^*)^W_+$ denote the ideal of invariants without constant term, and consider the coinvariant algebra $\mathrm{Sym}(V^*)/\mathrm{Sym}(V^*)^W_+$. It follows from the results in [**39**,**104**] that this algebra contains exactly k copies of any irreducible W-representation U of dimension k. The degrees of the homogeneous components in which these k copies of U occur are called the *U-exponents of W*, and they are denoted by $e_1(U), e_2(U), \ldots, e_k(U)$. Now it follows from [**73**, Section 4.1] that the degrees of W satisfy $d_i = e_i(V) + 1$ for $i \in \{1, 2, \ldots, n\}$. Moreover, we can define the *codegrees $d_1^*, d_2^*, \ldots, d_n^*$ of W* by $d_i^* = e_1(V^*) - 1$ for $i \in \{1, 2, \ldots, n\}$, see also [**73**, Definition 10.27]. If we assume that the degrees of W are indexed in nondecreasing order and the codegrees of W are indexed in nonincreasing order, then a complex reflection group is *well-generated* if

$$d_i + d_i^* = d_n, \tag{4.1}$$

for all $i \in \{1, 2, \ldots, n\}$. Equivalently, W is well-generated if it can be generated by n reflections, where n is the rank of W, see [**87**]. We can conclude from [**87**, Table 2] that there are three infinite families of well-generated complex reflection groups, namely $G(1, 1, n)$ for $n \geq 2$, as well as $G(d, 1, n)$ and $G(d, d, n)$ for $n, d \geq 2$. It follows further that 26 of the 34 exceptional irreducible finite complex reflection groups are well-generated. We list them in Section 4.4.2.

4.2.2. Coxeter Elements and Noncrossing Partitions. Now we want to generalize the concept of a Coxeter element to well-generated complex reflection groups. There are many different definitions in the literature, and we follow the one given in [**38**]. We say that $w \in W$ is *ζ-regular* if w has an eigenvector to an eigenvalue ζ that does not lie in one of the reflection hyperplanes of W, and the multiplicative order of w is a *regular number for W*. The following result is due to Lehrer and Springer.

Theorem 4.2.2 ([**72**, Theorem C])

Let d be any integer and let W be a complex reflection group. Then, d is a regular number for W if and only if it divides as many degrees of W as it divides codegrees of W.

While the original proof of this result was case-by-case, a uniform proof was given later by Lehrer and Michel in [**71**]. Theorem 4.2.2 and (4.1) imply that d_n is a regular number for every well-generated complex reflection group. In this case we call d_n the *Coxeter number of W* and write h instead of d_n. It follows that for any primitive h-th root of unity ζ there exists a regular element $\gamma_\zeta \in W$ with eigenvalue ζ, see [**73**, Remark 11.23]. We call such an element a *Coxeter element of W*.

Remark 4.2.3

Let W be a real reflection group, and let T denote the set of reflections of W. In this case the definition of a Coxeter element of W given here coincides with the one given in Section 1.2.4 for some Coxeter system (W, S) with $S \subseteq T$, see [**99**, Table 1].

For the rest of this chapter, unless otherwise stated, W denotes an irreducible well-generated complex reflection group, T denotes the set of all reflections of W, ε denotes the identity of W, and γ denotes a Coxeter element of W. Since T is a generating set of W, we can generalize the notions of absolute length, reduced T-decomposition and absolute order, defined in Sections 1.2.4 and 1.2.5, verbatim from finite Coxeter groups to complex reflection groups. Let $NC_W(\gamma) = \{w \in W \mid w \leq_T \gamma\}$, and call the poset $\mathcal{NC}_W(\gamma) = \big(NC_W(\gamma), \leq_T\big)$ the *lattice of W-noncrossing partitions*. The fact that $\mathcal{NC}_W(\gamma)$ is indeed a lattice for any choice of Coxeter element γ has been proven in a series of papers by different authors, see [**12**,**14**,**31**,**33**]. We remark that Brady and Watt provide a uniform proof of the lattice property of $\mathcal{NC}_W(\gamma)$ in [**34**] for the case where W is a Coxeter group, while the general result still relies on a case-by-case analysis. It follows immediately from the definition of the absolute order that $\mathcal{NC}_W(\gamma)$ is graded. Moreover, this lattice does not depend on the choice of γ.

PROPOSITION 4.2.4 ([99, Corollary 1.6])

Let W be an irreducible well-generated complex reflection group, and let γ and γ' be two Coxeter elements. Then the two posets $\mathcal{NC}_W(\gamma)$ and $\mathcal{NC}_W(\gamma')$ are isomorphic.

DISCLAIMER 4.2.5

If W is reducible, then we can write $W \cong W_1 \times W_2 \times \cdots W_k$, where $W_1, W_2, \ldots, W_k$ are irreducible complex reflection groups. Moreover, if W is well-generated, then so is W_i for each $i \in \{1, 2, \ldots, k\}$. For $i \in \{1, 2, \ldots, k\}$ let γ_i be a Coxeter element of W_i. Then we can consider the product $\gamma = \gamma_1\gamma_2 \cdots \gamma_k$ as a substitute of a Coxeter element of W. It follows immediately that

$$\mathcal{NC}_W(\gamma) \cong \mathcal{NC}_{W_1}(\gamma_1) \times \mathcal{NC}_{W_2}(\gamma_2) \times \cdots \times \mathcal{NC}_{W_k}(\gamma_k).$$

Since we want to study the topology of the lattices $\mathcal{NC}_W$ it is sufficient to understand the topology of its irreducible components, and thus it is sufficient to investigate the irreducible well-generated complex reflection groups.

In [1] ARMSTRONG defined a generalization of $\mathcal{NC}_W(\gamma)$ as follows: consider the set

$$NC_W^{(m)}(\gamma) = \left\{ (w_0; w_1, \ldots, w_m) \in NC_W^{m+1}(\gamma) \mid \gamma = w_0 w_1 \cdots w_m \text{ and } \sum_{i=0}^{m} \ell_T(w_i) = \ell_T(\gamma) \right\},$$

and define a partial order on $NC_W^{(m)}(\gamma)$ by

$$(u_0; u_1, \ldots, u_m) \leq_T (v_0; v_1, \ldots, v_m) \quad \text{if and only if} \quad u_i \geq_T v_i, \text{ for all } i \in \{1, 2, \ldots, m\}.$$

The corresponding poset $\mathcal{NC}_W^{(m)}(\gamma) = \big(NC_W^{(m)}(\gamma), \leq\big)$ is called the *poset of m-divisible W-noncrossing partitions*. It is in general a join-semilattice, since it has no least element, and it is graded by the rank function $\mathrm{rk}(w_0; w_1, \ldots, w_m) = \ell_T(w_0)$. If $m = 1$, then we obtain the previously introduced lattice of W-noncrossing partitions. Again in view of Proposition 4.2.4 the structure of the poset $\mathcal{NC}_W^{(m)}(\gamma)$ does not depend on the particular choice of Coxeter element γ. Thus, unless we explicitly need to consider $\mathcal{NC}_W(\gamma)$ for a fixed Coxeter element γ, we drop the Coxeter element from the notation and write $\mathcal{NC}_W$ instead. Although ARMSTRONG originally considered only Coxeter groups, the same construction can be carried out in the general setting of well-generated complex reflection groups, see [16].

Another remarkable property of $\mathcal{NC}_W^{(m)}$ is that its cardinality is given by the W-Fuß-Catalan number.

THEOREM 4.2.6 ([1,7,13,14,47,70,98])

Let W be a well-generated complex reflection group, let $d_1, d_2, \ldots, d_n$ denote its degrees in nondecreasing order and let h denote its Coxeter number. For every $m > 0$ we have

$$\big|NC_W^{(m)}\big| = \prod_{i=1}^{n} \frac{mh + d_i}{d_i} = Cat^{(m)}(W).$$

Remark 4.2.7

If $W = G(1,1,n)$, or equivalently, if $W = A_{n-1}$ is the symmetric group, then the posets $\mathcal{NC}_W^{(m)}$ coincide with the classical posets of m-divisible noncrossing set partitions $\mathcal{NC}_n^{(m)}$ of Edelman. If $m = 1$, then we obtain the lattice of noncrossing set partitions $\mathcal{NC}_n$ of Kreweras.

Example 4.2.8

Consider the well-generated complex reflection group $G(5,5,3)$ of rank 3. It follows for instance from [88, Table 2] that the degrees of $G(5,5,3)$ are $d_1 = 3, d_2 = 5$ and $d_3 = 10$, which yields the Coxeter number $h = 10$. Hence the lattice $\mathcal{NC}_{G(5,5,3)}$ has 26 elements, and it is depicted in Figure 43 on page 121. The labels are explained in Example 4.4.9.

4.3. Basic Properties

First we observe that the set of reflections of a complex reflection group is closed under conjugation.

Lemma 4.3.1 ([73, Lemma 1.9])

For every $t \in T$ and every $w \in W$, we have $w^{-1}tw \in T$.

For $w \in W$ define the *fixed space of w* by $\mathrm{Fix}(w) = \{\mathbf{v} \in V \mid w(\mathbf{v}) = \mathbf{v}\}$. Let $A \subseteq V$ be a subspace of V. The maximal subgroup W' of W that fixes A pointwise is called a *parabolic subgroup of W*. We have the following result.

Theorem 4.3.2 ([117, Theorem 1.5])

Every parabolic subgroup of a complex reflection group is a complex reflection group in its own right.

By inspecting the classification of complex reflection groups we obtain the following corollary.

Corollary 4.3.3 ([13, Lemma 2.7])

Every parabolic subgroup of a well-generated complex reflection group is again well-generated.

In fact we can say a bit more.

Proposition 4.3.4 ([100, Proposition 6.3(i),(ii)])

Let W be a well-generated complex reflection group, and let $w \in W$. The following are equivalent:

(i) w is a Coxeter element in a parabolic subgroup of W; and
(ii) there is a Coxeter element $\gamma_w \in W$ such that $w \leq_T \gamma_w$.

We call w a *parabolic Coxeter element* if it satisfies one of the properties stated in Proposition 4.3.4. Now fix some Coxeter element $\gamma \in W$, and let $T_\gamma = \{t \in T \mid t \leq_T \gamma\}$ denote the set of reflections of W that are contained in $\mathcal{NC}_W(\gamma)$. Let us define an edge-labeling of $\mathcal{NC}_W(\gamma)$ by

$$\lambda_\gamma : \mathcal{E}\big(\mathcal{NC}_W(\gamma)\big) \to T_\gamma, \quad (u,v) \mapsto u^{-1}v. \tag{4.2}$$

This labeling arises quite naturally from the definition of the absolute order, and it has some nice properties. These properties are well-known to the community, but we give their proofs for the sake of self-containedness.

LEMMA 4.3.5

Let $u, v \in NC_W(\gamma)$ with $u \leq_T v$. A product $t_1 t_2 \cdots t_k$ is a reduced T-decomposition of $u^{-1}v$ if and only if there exists a maximal chain $C : u = x_0 \lessdot_T x_1 \lessdot_T \cdots \lessdot_T x_k = v$ in $\mathcal{NC}_W(\gamma)$ with $\lambda(C) = (t_1, t_2, \ldots, t_k)$.

PROOF. Let $C : u = x_0 \lessdot_T x_1 \lessdot_T \cdots \lessdot_T x_k = v$ be a maximal chain in $[u, v]$ with $\lambda(C) = (t_1, t_2, \ldots, t_k)$. Since $\mathcal{NC}_W(\gamma)$ is graded, we conclude that $\ell_T(u^{-1}v) = k$. By definition of λ_γ, we obtain $x_{i-1}^{-1}x_i = t_i$ for all $i \in \{1, 2, \ldots, k\}$. Thus

$$t_1 t_2 \cdots t_k = x_0^{-1} x_1 x_1^{-1} x_2 \cdots x_{k-1}^{-1} x_k = u^{-1} v$$

as desired.

Conversely let $t_1 t_2 \cdots t_k$ be a reduced T-decomposition of $u^{-1}v$. Define $x_0 = u$ and $x_i = u t_1 t_2 \cdots t_i$ for $i \in \{1, 2, \ldots, k\}$. It follows that $x_k = v$ and

$$x_{i-1}^{-1} x_i = t_{i-1}^{-1} t_{i-2}^{-1} \cdots t_1^{-1} u^{-1} u t_1 t_2 \cdots t_{i-1} t_i = t_i$$

This implies $x_{i-1} \lessdot_T x_i$ and $\lambda_\gamma(x_{i-1}, x_i) = t_i$ for all $i \in \{1, 2, \ldots, k\}$. Hence $C : u = x_0 \lessdot_T x_1 \lessdot_T \cdots \lessdot_T x_k = v$ is a maximal chain in $[u, v]$ with $\lambda(C) = (t_1, t_2, \ldots, t_k)$. □

LEMMA 4.3.6

Let $u, v \in NC_W(\gamma)$ with $u \leq_T v$. The poset isomorphism $f : [\varepsilon, u^{-1}v] \to [u, v]$ given by $f(x) = ux$ satisfies $\lambda_\gamma(x, y) = \lambda_\gamma(f(x), f(y))$ for all cover relations (x, y) in $[\varepsilon, u^{-1}v]$.

PROOF. Let $x, y \in NC_W(\gamma)$ with $x \lessdot_T y \leq_T u^{-1}v$. By definition, there exists some $t \in T_\gamma$ with $y = xt$. Then we have $f(y) = uy = uxt = f(x)t$ and thus $f(x) \lessdot_T f(y)$, and vice versa. Hence f is indeed a poset isomorphism, and it satisfies

$$\lambda_\gamma(x, y) = x^{-1} y = x^{-1} u^{-1} u y = (ux)^{-1} u y = \lambda_\gamma(ux, uy) = \lambda_\gamma(f(x), f(y)).$$

□

Furthermore the reduced T-decompositions of some $w \in W$ can be obtained from one another by repeated "shifting of letters". This is made precise in the following lemma, which first appeared in [1, Lemma 2.5.1] for Coxeter groups.

LEMMA 4.3.7

If $w = t_1 t_2 \cdots t_k$ is a reduced T-decomposition, then so are

(4.3) $$t_1 t_2 \cdots t_{i-2} t_i (t_i^{-1} t_{i-1} t_i) t_{i+1} t_{i+2} \cdots t_k,$$

for $i \in \{2, 3, \ldots, k\}$ and

(4.4) $$t_1 t_2 \cdots t_{i-1} (t_i t_{i+1} t_i^{-1}) t_i t_{i+2} t_{i+3} \cdots t_k,$$

for all $i \in \{1, 2, \ldots, k-1\}$.

Proof. Since $w = t_1t_2\cdots t_k$ is a reduced T-decomposition, it follows that $\ell_T(w) = k$. Clearly, for suitable i, it follows that

$$w = t_1t_2\cdots t_{i-2}t_i(t_i^{-1}t_{i-1}t_i)t_{i+1}t_{i+2}\cdots t_k,$$

and

$$w = t_1t_2\cdots t_{i-1}(t_it_{i+1}t_i^{-1})t_it_{i+2}t_{i+3}\cdots t_k,$$

hence both are T-decompositions of w. Lemma 4.3.1 implies that $t_i^{-1}t_{i-1}t_i$ and $t_it_{i+1}t_i^{-1}$ are both reflections themselves, and hence $\ell_T(t_i^{-1}t_{i-1}t_i) = \ell_T(t_it_{i+1}t_i^{-1}) = 1$. It follows that both T-decompositions of w are reduced. $\square$

We notice that in (4.3) the letter t_i appears in the $(i-1)$-st position, and in (4.4) it appears in the $(i+1)$-st position. Hence it is justified to refer to this procedure as *left* or *right-shifting of w*, respectively. This useful property of reduced T-decompositions has some effect on the label sequences of closed intervals of $\mathcal{NC}_W(\gamma)$. The next two results are straightforward generalizations of the corresponding results that Athanasiadis, Brady and Watt obtained for Coxeter groups, see [6, Lemma 3.6 and Theorem 3.5(i)], and the proofs given here are modeled after the corresponding proofs in [6]. For some closed interval $[u,v]$ in $\mathcal{NC}_W(\gamma)$ we abbreviate

$$\lambda_\gamma\big([u,v]\big) = \big\{\lambda_\gamma(C) \mid C \text{ is a maximal chain in } [u,v]\big\}.$$

Lemma 4.3.8

Let $[u,v]$ be a non-singleton interval in $\mathcal{NC}_W(\gamma)$.

(i) If $[u,v]$ has rank two and $(r,t) \in \lambda_\gamma\big([u,v]\big)$, then there exists some $r' \in T_\gamma$ with $(r',t) \in \lambda_\gamma\big([u,v]\big)$.

(ii) If $t \in T_\gamma$ appears as a label in some element of $\lambda_\gamma\big([u,v]\big)$, then $t = \lambda_\gamma(u,u')$ for some cover relation (u,u') in $[u,v]$.

(iii) Suppose that all reflections in W have order 2. The reflections appearing as the labels of some element of $\lambda_\gamma\big([u,v]\big)$ are pairwise distinct.

Proof. (i) Let $(r,t) \in \lambda_\gamma\big([u,v]\big)$. Lemma 4.3.5 implies that $u^{-1}v = rt$. Lemma 4.3.7 implies $u^{-1}v = tr'$ with $r' = t^{-1}rt$, and again Lemma 4.3.5 yields $(t,r') \in \lambda_\gamma\big([u,v]\big)$.

(ii) This follows from repeated application of (i), using the fact that $\mathcal{NC}_W(\gamma)$ is graded.

(iii) Let C be a maximal chain in $[u,v]$ with $\lambda_\gamma(C) = (t_1,t_2,\ldots,t_k)$. Suppose that there exist indices $i,j \in \{1,2,\ldots,k\}$ with $i < j$ such that $t_i = t_j$. Lemma 4.3.5 implies $u^{-1}v = t_1t_2\cdots t_k$. Repeated application of Lemma 4.3.7 yields $u^{-1}v = t_1t_2\cdots t_{i-1}t_it_jt'_{i+2}t'_{i+3}\cdots t'_k$. Since t_i and t_j have order 2 and $t_i = t_j$ they cancel, and it follows that $\ell_T(u^{-1}v) < k$, which is a contradiction. $\square$

Proposition 4.3.9

Let W be a well-generated complex reflection group in which every reflection has order 2, and let $\gamma \in W$ be a Coxeter element. For any total order on T_γ and any non-singleton interval $[u,v]$ in $\mathcal{NC}_W(\gamma)$ the lexicographically first maximal chain in $[u,v]$ is rising with respect to λ_γ.

Proof. Let $[u,v]$ be a non-singleton interval in $\mathcal{NC}_W(\gamma)$, and let $\prec$ be a total order on T_γ. We proceed by induction on $\ell_T(u^{-1}v)$. If $\ell_T(u^{-1}v) = 1$, then the claim is trivially true. Let $\ell_T(u^{-1}v) = k$, and suppose that the claim is true for all intervals $[\bar{u},\bar{v}]$ in $\mathcal{NC}_W(\gamma)$ with

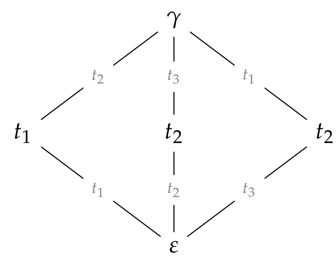


FIGURE 42. The lattice $\mathcal{NC}_{G(3,3,2)}$ with the edge-labeling from (4.2).

$\ell_T(\bar{u}^{-1}\bar{v}) < k$. It is easy to see that all cover relations (u, u') with $u' \leq_T v$ have different labels with respect to λ_γ. Now let $t = \min\{\lambda_\gamma(u, ut') \mid t' \in T_\gamma \text{ and } ut' \leq_T v\}$, where the minimum is taken with respect to $\prec$. Suppose that there is a chain in $[ut, v]$ having an edge labeled by some $r \in T_\gamma$ with $r \prec t$. Lemma 4.3.8(ii) implies that $u \lessdot_T ur \leq_T v$, contradicting the choice of t. Moreover, Lemma 4.3.8(iii) implies that t does not occur as a label in $\lambda_\gamma\big([ut, v]\big)$. By induction hypothesis the lexicographically first maximal chain in $[ut, v]$ is rising, and in view of the previous reasoning we can append this chain to the edge (u, ut). This implies that the lexicographically first maximal chain in $[u, v]$ is rising. $\square$

REMARK 4.3.10

Proposition 4.3.9 implies that in a well-generated reflection group W, in which all reflections have order 2, the lexicographically first maximal chain in $\mathcal{NC}_W(\gamma)$ is rising with respect to λ_γ and any total order on T_γ. However, in general there exist more than just this one rising maximal chain.

Consider for instance $W = G(3,3,2)$, namely the dihedral group of order 6, or equivalently the symmetric group on $\{1,2,3\}$. Since W is a Coxeter group, every reflection has order 2 and Proposition 4.3.9 can be applied. Let t_1, t_2 and t_3 denote its reflections. We can interpret these reflections as transpositions, say $t_1 = (1\ 2), t_2 = (2\ 3)$ and $t_3 = (1\ 3)$. Consider the Coxeter element $\gamma = t_1t_2$. The corresponding lattice $\mathcal{NC}_{G(3,3,2)}$ is shown in Figure 42, and the edges are labeled by λ_γ. We notice that among the six total orders of $T_\gamma = \{t_1, t_2, t_3\}$ exactly three make λ_γ an EL-labeling, namely

$$t_1 \prec t_3 \prec t_2, \quad t_2 \prec t_1 \prec t_3, \quad t_3 \prec t_2 \prec t_1,$$

while the other three total orders produce two increasing maximal chains.

4.4. Topological Properties

This section is dedicated to the proof of the main result of this chapter.

THEOREM 4.4.1

Let W be a well-generated complex reflection group, and let $\gamma \in W$ be a Coxeter element. Let $\overline{\mathcal{NC}}_W^{(m)}(\gamma)$ denote the lattice that is constructed from $\mathcal{NC}_W^{(m)}(\gamma)$ by adding a least element. Then, $\overline{\mathcal{NC}}_W^{(m)}(\gamma)$ is EL-shellable for all $m > 0$.

Again this result was proven earlier for some special cases. In the case where W is a Coxeter group, Theorem 4.4.1 is [6, Theorem 1.1] if $m = 1$, and it is [1, Theorem 3.7.2] if $m > 1$.

But more can be said. Recall that the *braid group* associated with a complex reflection group, denoted by $\mathfrak{B}(W)$, is the fundamental group of the complement of the reflection hyperplanes of W. We have the following result.

Proposition 4.4.2

For $d, n \geq 2$, we have $\mathcal{NC}_{G(d,1,n)} \cong \mathcal{NC}_{B_n}$. Moreover, we have $\mathcal{NC}_{G_{25}} \cong \mathcal{NC}_{A_3}, \mathcal{NC}_{G_{26}} \cong \mathcal{NC}_{B_3}$ and $\mathcal{NC}_{G_{32}} \cong \mathcal{NC}_{A_4}$.

Proof. It follows for instance from [35, Table 1] that for $d, n \geq 2$ we have $\mathfrak{B}\big(G(d,1,n)\big) \cong \mathfrak{B}(B_n)$, and moreover that $\mathfrak{B}(G_{25}) \cong \mathfrak{B}(A_3), \mathfrak{B}(G_{26}) \cong \mathfrak{B}(B_3)$, and $\mathfrak{B}(G_{32}) \cong \mathfrak{B}(A_4)$. In [13], Bessis showed that $\mathcal{NC}_W$ can be realized as a poset of so-called simple elements of $\mathfrak{B}(W)$. Since the braid groups in question are isomorphic, so are their simple elements, and the claim follows. □

Remark 4.4.3

Using the braid group perspective, we can generalize Proposition 4.3.9 to all well-generated complex reflection groups. It follows from [13, Theorem 2.2] that for every complex reflection group W there exists a complex reflection group W' in which all reflections have order 2 such that $\mathfrak{B}(W) \cong \mathfrak{B}(W')$. Thus it follows in this case that $\mathcal{NC}_W \cong \mathcal{NC}_{W'}$, and we can transfer the property that the lexicographically first maximal chain is rising with respect to any total order on the reflections with this isomorphism.

In view of Disclaimer 4.2.5 it suffices to prove Theorem 4.4.1 for the remaining irreducible well-generated complex reflection groups. More precisely, it remains to prove Theorem 4.4.1 for the groups $G(d,d,n)$, where $d, n \geq 3$, and for the exceptional well-generated complex reflection groups that are no Coxeter groups and that do not occur in Proposition 4.4.2. We give an explicit total order of T_γ for $G(d,d,n)$ that makes λ_γ an EL-labeling of $\mathcal{NC}_{G(d,d,n)}$ in Section 4.4.1, and we treat the exceptional groups by computer in Section 4.4.2. It follows from the proof of [1, Theorem 3.7.2] that once one has an EL-labeling of $\mathcal{NC}_W$ it is straightforward to construct an EL-labeling of $\mathcal{NC}_W^{(m)}$ out of it. We recall this construction in Section 4.4.3. We conclude this chapter by deriving results on the Möbius function of $\mathcal{NC}_W^{(m)}$ in Section 4.4.4.

4.4.1. The Groups $G(d,d,n)$ for $d, n \geq 3$. In general the complex reflection groups $G(d,e,n)$ have a wreath product structure, which allows for the representation of these groups in terms of monomial matrices, see [73, Chapter 2.2]. We call this representation the *standard monomial representation of $G(d,e,n)$*.

In the following we explicitly describe this representation for the groups $G(d,d,n)$ for $d, n \geq 3$, and accompany these explanations by the running example of the group $G(5,5,3)$. In this representation the elements of $G(d,d,n)$ are monomial $(n \times n)$-matrices, where the nonzero entries are d-th roots of unity and where the product of all nonzero entries is 1. These matrices act as permutation matrices on the set

$$\big\{1^{(0)}, 2^{(0)}, \ldots, n^{(0)}, 1^{(1)}, 2^{(1)}, \ldots, n^{(2)}, 1^{(3)}, \ldots, n^{(d-1)}\big\} \tag{4.5}$$

of integers with d colors. For all $k \in \{1, 2, \ldots, n\}$ and $s \in \{0, 1, \ldots, d-1\}$, we identify the colored integer $k^{(s)}$ with the vector $\zeta_d^s \cdot \mathbf{e}_k$, where $\mathbf{e}_k$ denotes the k-th unit vector of $\mathbb{C}^n$ and where $\zeta_d = e^{2\pi\sqrt{-1}/d}$ is a primitive d-th root of unity. Hence $G(d,d,n)$ is isomorphic to a subgroup of the group of permutations of the set (4.5), and it consists of elements w satisfying

$$w\big(k^{(s)}\big) = \sigma(k)^{(s+t_k)},$$

for some permutation σ of $\{1,2,\ldots,n\}$ and some $t_k \in \mathbb{Z}$ that depends on w, and where the addition in the superscript is understood modulo d. The numbers t_k have to satisfy the property

$$\sum_{k=1}^{n} t_k \equiv 0 \pmod{d}.$$

This allows us to represent the elements of $G(d,d,n)$ in a permutation-like fashion by

$$\begin{pmatrix} 1^{(0)} & 2^{(0)} & \ldots & n^{(0)} \\ \sigma(1)^{(t_1)} & \sigma(2)^{(t_2)} & \ldots & \sigma(n)^{(t_n)} \end{pmatrix}.$$

We can thus decompose the elements of $G(d,d,n)$ into cycles, and we use the following abbreviations:

$$\begin{aligned}\big(\big(k_1^{(t_1)}\ k_2^{(t_2)}\ \ldots\ k_r^{(t_r)}\big)\big) = {}& \big(k_1^{(t_1)}\ k_2^{(t_2)}\ \ldots\ k_r^{(t_r)}\big) \\ & \big(k_1^{(t_1+1)}\ k_2^{(t_2+1)}\ \ldots\ k_r^{(t_r+1)}\big)\cdots\big(k_1^{(t_1+d-1)}\ k_2^{(t_2+d-1)}\ \ldots\ k_r^{(t_r+d-1)}\big)\end{aligned}$$

and

$$\begin{aligned}\big[k_1^{(t_1)}\ k_2^{(t_2)}\ \ldots\ k_r^{(t_r)}\big]_s = {}& \big(k_1^{(t_1)}\ k_2^{(t_2)}\ \ldots\ k_r^{(t_r)} \\ & k_1^{(t_1+s)}\ k_2^{(t_2+s)}\ \ldots\ k_r^{(t_r+s)}\ \ldots\ k_1^{(t_1+(d-1)s)}\ k_2^{(t_2+(d-1)s)}\ \ldots\ k_r^{(t_r+(d-1)s)}\big).\end{aligned}$$

If $s = 1$, then we usually write $\big[k_1^{(t_1)}\ k_2^{(t_2)}\ \ldots\ k_r^{(t_r)}\big]$ instead of $\big[k_1^{(t_1)}\ k_2^{(t_2)}\ \ldots\ k_r^{(t_r)}\big]_1$. It is immediate that every element of $G(d,d,n)$ can be decomposed into "cycles" of these two forms. Usually we write $w \in G(d,d,n)$ in this "cycle notation", and we denote by $\varphi(w)$ the corresponding monomial matrix.

EXAMPLE 4.4.4

Consider $G(5,5,3)$ and consider the element

$$w = \begin{pmatrix} 1^{(0)} & 2^{(0)} & 3^{(0)} \\ 2^{(2)} & 1^{(1)} & 3^{(2)} \end{pmatrix}.$$

The corresponding monomial matrix is

$$\varphi(w) = \begin{pmatrix} 0 & \zeta_5 & 0 \\ \zeta_5^2 & 0 & 0 \\ 0 & 0 & \zeta_5^2 \end{pmatrix}.$$

Hence w acts on the set

$$\big\{1^{(0)},2^{(0)},3^{(0)},1^{(1)},2^{(1)},3^{(1)},1^{(2)},2^{(2)},3^{(2)},1^{(3)},2^{(3)},3^{(3)},1^{(4)},2^{(4)},3^{(4)}\big\}$$

as follows:

$$\begin{aligned} w\big(1^{(0)}\big) &= 2^{(2)}, & w\big(2^{(0)}\big) &= 1^{(1)}, & w\big(3^{(0)}\big) &= 3^{(2)}, \\ w\big(1^{(1)}\big) &= 2^{(3)}, & w\big(2^{(1)}\big) &= 1^{(2)}, & w\big(3^{(1)}\big) &= 3^{(3)}, \\ w\big(1^{(2)}\big) &= 2^{(4)}, & w\big(2^{(2)}\big) &= 1^{(3)}, & w\big(3^{(2)}\big) &= 3^{(4)}, \\ w\big(1^{(3)}\big) &= 2^{(0)}, & w\big(2^{(3)}\big) &= 1^{(4)}, & w\big(3^{(3)}\big) &= 3^{(0)}, \end{aligned}$$

$$w\big(1^{(4)}\big) = 2^{(1)},\quad w\big(2^{(4)}\big) = 1^{(0)},\quad w\big(3^{(4)}\big) = 3^{(1)}.$$

The cycle decomposition of w is

$$w = \big[1^{(0)}\ 2^{(2)}\big]_3 \big[3^{(0)}\big]_2.$$

It is our goal to show that the labeling defined in (4.2) is an EL-labeling of $\mathcal{NC}_{G(d,d,n)}$ with respect to a suitable total order on the reflections of $G(d,d,n)$. In order to do so, it is necessary to understand what the reflections of $G(d,d,n)$ look like.

Proposition 4.4.5 ([73, Proposition 2.9])

The group $G(d,d,n)$ contains $d\binom{n}{2}$ reflections, and the order of every reflection is 2.

Since the reflections of $G(d,d,n)$ are unitary involutions that fix a space of codimension 1, it follows immediately that

$$T = \Big\{ \big(\big(a^{(0)}\ b^{(s)}\big)\big) \mid 1 \leq a < b \leq n, 0 \leq s < d \Big\}. \tag{4.6}$$

Let us emphasize a certain subset of T, namely the reflections

$$\big(\big(1^{(0)}\ 2^{(0)}\big)\big), \big(\big(2^{(0)}\ 3^{(0)}\big)\big), \cdots, \big(\big((n{-}1)^{(0)}\ n^{(0)}\big)\big), \big(\big((n{-}1)^{(0)}\ n^{(1)}\big)\big),$$

and we call them the *simple reflections of $G(d,d,n)$*. In what follows, we use the abbreviations $s_i = \big(\big(i^{(0)}\ (i{+}1)^{(0)}\big)\big)$ for $i \in \{1,2,\ldots,n-1\}$ and $s_n = \big(\big((n{-}1)^{(0)}\ n^{(1)}\big)\big)$. The product $\gamma = s_1 s_2 \cdots s_n$ is the group element

$$\gamma = \big[1^{(0)}\ 2^{(0)}\ \ldots\ (n{-}1)^{(0)}\big] \big[n^{(0)}\big]_{d-1}, \tag{4.7}$$

which can be represented by the monomial matrix

$$\varphi(\gamma) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \zeta_d & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \zeta_d^{d-1} \end{pmatrix}. \tag{4.8}$$

Recall for instance from [87, Table 2] that the degrees of $G(d,d,n)$ are

$$d, 2d, \ldots, (n-1)d, n,$$

and it follows that the Coxeter number of $G(d,d,n)$ is $h = (n-1)d$. We can check that ζ_h is an eigenvalue of $\varphi(\gamma)$, and an eigenvector of $\varphi(\gamma)$ to ζ_h is for instance

$$\mathbf{v} = \begin{pmatrix} \zeta_h^{n-1} & \zeta_h^{n-2} & \ldots & \zeta_h & 0 \end{pmatrix}^{\mathsf{T}}, \tag{4.9}$$

where "T" denotes the transposition of vectors. The reflection hyperplanes of $G(d,d,n)$ (in standard monomial representation) are given by the equations

$$x_i = \zeta_d^s x_j,\quad \text{for } 1 \leq i < j \leq n \text{ and } 0 \leq s < d.$$

Hence the vector $\mathbf{v}$ from (4.9) is indeed ζ_h-regular, which makes γ a Coxeter element of $G(d,d,n)$. For later use, we refer to the reduced T-decomposition

$$\gamma = \big(\big(1^{(0)}\ 2^{(0)}\big)\big)\big(\big(2^{(0)}\ 3^{(0)}\big)\big) \cdots \big(\big((n{-}1)^{(0)}\ n^{(0)}\big)\big)\big(\big((n{-}1)^{(0)}\ n^{(1)}\big)\big) \tag{4.10}$$

as the *simple decomposition of* γ. In the remainder of this section, we will always consider the Coxeter element γ from (4.7).

Remark 4.4.6

If we consider the subword $\bar{\gamma} = \big(\!\big(1^{(0)}\ 2^{(0)}\ \ldots\ n^{(0)}\big)\!\big) = \gamma s_n$, then we obtain a reduced T-decomposition

$$\bar{\gamma} = \big(\!\big(1^{(0)}\ 2^{(0)}\big)\!\big)\big(\!\big(2^{(0)}\ 3^{(0)}\big)\!\big)\cdots\big(\!\big((n-1)^{(0)}\ n^{(0)}\big)\!\big), \tag{4.11}$$

which we will refer to as the *simple decomposition of* $\bar{\gamma}$. More precisely, $\bar{\gamma}$ is a Coxeter element in a parabolic subgroup of $G(d,d,n)$ that is isomorphic to $G(1,1,n)$ and has rank $n-1$. We call the reflections $s_1, s_2, \ldots, s_{n-1}$ the *simple reflections of* $G(1,1,n)$. Indeed, there is an obvious bijection from the set $\{s_1, s_2, \ldots, s_{n-1}\}$ to the set of adjacent transpositions of A_{n-1}.

The groups $G(d,d,n)$ are well-behaved in the sense that the absolute length of elements in $NC_{G(d,d,n)}$ is the codimension of their fixed space.

Lemma 4.4.7 ([14, Lemma 4.1])

For $w \in NC_{G(d,d,n)}$ we have $\ell_T(w) = n - \dim \mathrm{Fix}(w)$.

The next proposition characterizes the set T_γ for the Coxeter element γ from (4.7).

Proposition 4.4.8

Let γ be the Coxeter element of $G(d,d,n)$ from (4.7). We have

$$T_\gamma = \Big\{\big(\!\big(a^{(0)}\ b^{(s)}\big)\!\big) \mid 1 \leq a < b < n, s \in \{0, d-1\}\Big\} \cup \Big\{\big(\!\big(a^{(0)}\ n^{(s)}\big)\!\big) \mid 1 \leq a < n, 0 \leq s < d\Big\}.$$

Proof. Proposition 4.4.5 implies that the reflections of $G(d,d,n)$ are involutions, and it follows from the definition that $\ell_T(t) = 1$ if and only if $t \in T$. Hence Lemma 4.4.7 implies that $t \leq_T \gamma$ if and only if $\dim \mathrm{Fix}(t\gamma) = 1$, where $\mathrm{Fix}(w) = \{\mathbf{v} \in \mathbb{C}^n \mid w\mathbf{v} = \mathbf{v}\}$. For an arbitrary vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)^{\mathsf{T}} \in \mathbb{C}^n$, we have

$$\mathbf{v}' = \gamma\mathbf{v} = \Big(\zeta_d v_{n-1}, v_1, v_2, \ldots, v_{n-2}, \zeta_d^{d-1} v_n\Big)^{\mathsf{T}}. \tag{4.12}$$

In what follows, we determine the dimension of $\mathrm{Fix}(t\gamma)$ for $t \in T$. We distinguish three cases:

(i) Let $t = \big(\!\big(a^{(0)}\ b^{(s)}\big)\!\big)$, where $1 \leq a < b < n$ and $0 \leq s < d$. We obtain

$$t\mathbf{v}' = \Big(\zeta_d v_{n-1}, v_1, \ldots, v_{a-2}, \zeta_d^{d-s} v_{b-1}, v_a, \ldots, v_{b-2}, \zeta_d^{s} v_{a-1}, v_b, \ldots, v_{n-2}, \zeta_d^{d-1} v_n\Big)^{\mathsf{T}}.$$

Thus $\mathrm{Fix}(t\gamma)$ is given by the following system of linear equations:

$$\begin{array}{lllll} v_1 = \zeta_d v_{n-1}, & v_2 = v_1, & v_3 = v_2, & \ldots, & v_{a-1} = v_{a-2}, \\ v_a = \zeta_d^{d-s} v_{b-1}, & v_{a+1} = v_a, & v_{a+2} = v_{a+1}, & \ldots, & v_{b-1} = v_{b-2}, \\ v_b = \zeta_d^{s} v_{a-1}, & v_{b+1} = v_b, & v_{b+2} = v_{b+1}, & \ldots, & v_{n-1} = v_{n-2}, \\ v_n = \zeta_d^{d-1} v_n. & & & & \end{array}$$

If we put these equations together, then we obtain

$$\begin{aligned}\zeta_d^{s+1}v_{n-1} &= \zeta_d^s v_1 = \cdots = \zeta_d^s v_{a-1} = v_b = v_{b+1} = \cdots = v_{n-1},\\ \zeta_d^{d-s}v_{b-1} &= v_a = v_{a+1} = \cdots = v_{b-1},\\ \zeta_d^{d-1}v_n &= v_n.\end{aligned}$$

The first line has a nontrivial solution only if $s = d-1$ (which forces the components in lines 2 and 3 to be zero) and hence $\dim \mathrm{Fix}(t\gamma) = 1$. Similarly the second line has a nontrivial solution only if $s = 0$ (which forces the components in lines 1 and 3 to be zero) and hence $\dim \mathrm{Fix}(t\gamma) = 1$. Thus in these two cases we obtain $t \leq_T \gamma$. Every other value of s forces all components to be zero, which implies $\dim \mathrm{Fix}(t\gamma) = 0$ and hence $t \not\leq_T \gamma$.

(ii) Let $t = \big((1^{(0)}\ n^{(s)})\big)$, where $0 \leq s < d$. We obtain

$$t\mathbf{v}' = \left(\zeta_d^{d-s-1}v_n, v_1, v_2, \ldots, v_{n-2}, \zeta_d^{s+1}v_{n-1}\right)^{\mathsf{T}},$$

and analogously to (i) we see that $\dim \mathrm{Fix}(t\gamma) = 1$, which implies $t \leq_T \gamma$.

(iii) Let $t = \big((a^{(0)}\ n^{(s)})\big)$, where $1 < a < n$ and $0 \leq s < d$. We obtain

$$t\mathbf{v}' = \left(\zeta_d v_{n-1}, v_1, v_2, \ldots, v_{a-2}, \zeta_d^{d-s-1}v_n, v_a, v_{a+1}, \ldots, v_{n-2}, \zeta_d^s v_{a-1}\right)^{\mathsf{T}},$$

and analogously to (i) we see that $\dim \mathrm{Fix}(t\gamma) = 1$, which implies $t \leq_T \gamma$. $\square$

Proposition 4.4.5 states that all reflections of $G(d,d,n)$ have order 2. Hence we can apply Proposition 4.3.9, and we conclude that the lexicographically first maximal chain in $\mathcal{NC}_{G(d,d,n)}(\gamma)$ with respect to any total order of T_γ is rising. However, Remark 4.3.10 implies that not all total orders of T_γ make λ_γ an EL-labeling. We show in the remainder of this section that the following total order of T_γ works nicely.

$$\begin{array}{rcccccc} & \big((1^{(0)}\ 2^{(0)})\big) & \prec_\gamma & \big((1^{(0)}\ 3^{(0)})\big) & \prec_\gamma \cdots \prec_\gamma & \big((1^{(0)}\ (n{-}1)^{(0)})\big)\\ & & \prec_\gamma & \big((2^{(0)}\ 3^{(0)})\big) & \prec_\gamma \cdots \prec_\gamma & \big((2^{(0)}\ (n{-}1)^{(0)})\big)\\ & & \prec_\gamma & \big((3^{(0)}\ 4^{(0)})\big) & \prec_\gamma \cdots \prec_\gamma & \big(((n{-}2)^{(0)}\ (n{-}1)^{(0)})\big)\\ \prec_\gamma & \big((1^{(0)}\ n^{(0)})\big) & \prec_\gamma & \big((1^{(0)}\ n^{(d-1)})\big) & \prec_\gamma \cdots \prec_\gamma & \big((1^{(0)}\ n^{(1)})\big)\\ & & \prec_\gamma & \big((1^{(0)}\ 2^{(d-1)})\big) & \prec_\gamma \cdots \prec_\gamma & \big((1^{(0)}\ (n{-}1)^{(d-1)})\big)\\ \prec_\gamma & \big((2^{(0)}\ n^{(0)})\big) & \prec_\gamma & \big((2^{(0)}\ n^{(d-1)})\big) & \prec_\gamma \cdots \prec_\gamma & \big((2^{(0)}\ n^{(1)})\big)\\ & & \prec_\gamma & \big((2^{(0)}\ 3^{(d-1)})\big) & \prec_\gamma \cdots \prec_\gamma & \big((2^{(0)}\ (n{-}1)^{(d-1)})\big)\\ \prec_\gamma & \big((3^{(0)}\ n^{(0)})\big) & \prec_\gamma & \big((3^{(0)}\ n^{(d-1)})\big) & \prec_\gamma \cdots \prec_\gamma & \big(((n{-}1)^{(0)}\ n^{(1)})\big).\end{array} \tag{4.13}$$

EXAMPLE 4.4.9

Consider again $G(5,5,3)$. The total order of T_γ defined in (4.13) is:

$$\begin{aligned}\big((1^{(0)}\ 2^{(0)})\big) &\prec_\gamma \big((1^{(0)}\ 3^{(0)})\big) \prec_\gamma \big((1^{(0)}\ 3^{(4)})\big) \prec_\gamma \big((1^{(0)}\ 3^{(3)})\big) \prec_\gamma \big((1^{(0)}\ 3^{(2)})\big)\\ &\prec_\gamma \big((1^{(0)}\ 3^{(1)})\big) \prec_\gamma \big((1^{(0)}\ 2^{(4)})\big) \prec_\gamma \big((2^{(0)}\ 3^{(0)})\big) \prec_\gamma \big((2^{(0)}\ 3^{(4)})\big)\\ &\prec_\gamma \big((2^{(0)}\ 3^{(3)})\big) \prec_\gamma \big((2^{(0)}\ 3^{(2)})\big) \prec_\gamma \big((2^{(0)}\ 3^{(1)})\big).\end{aligned} \tag{4.14}$$

Figure 43 shows the lattice $\mathcal{NC}_{G(5,5,3)}(\gamma)$, and the integer labels correspond to positions in the total order on T_γ, for instance the label 3 represents $\big(\big(1^{(0)}\;3^{(4)}\big)\big)$. Likewise, the label $1{\cdot}9$ represents the product $\big(\big(1^{(0)}\;2^{(0)}\big)\big)\big(\big(2^{(0)}\;3^{(3)}\big)\big)$.

Remark 4.4.10

If $n=2$, then $G(d,d,2)$ is isomorphic to the Coxeter group $I_2(d)$, and the total order in (4.13) is precisely the total order given by Athanasiadis, Brady and Watt in [6, Example 3.2]. If $d=1$, then $G(1,1,n)$ is isomorphic to the Coxeter group A_{n-1}, and the total order in (4.13) is precisely the total order in [6, Example 3.3]. If $d=2$, then $G(2,2,n)$ is isomorphic to the Coxeter group D_n, and the total order in (4.13) is precisely the total order in [6, Example 3.4].

Our goal is to prove the following theorem.

Theorem 4.4.11

Let $d,n\geq 3$, let $\gamma\in G(d,d,n)$ be the Coxeter element defined in (4.7), and let T_γ denote the set of reflections of $G(d,d,n)$ that are contained in $NC_{G(d,d,n)}(\gamma)$. If T_γ is ordered as in (4.13), then the edge-labeling λ_γ of $\mathcal{NC}_{G(d,d,n)}(\gamma)$ defined in (4.2) is an EL-labeling.

The proof of this theorem consists of several steps, which we present separately in the following statements. In view of Lemma 4.3.5, we can (and will) use the terms "maximal chain in $[u,v]$" and "reduced T-decomposition of $u^{-1}v$" interchangeably. Hence a reduced T-decomposition of w is *rising* if the corresponding maximal chain in $[\varepsilon,w]$ is rising.

Lemma 4.4.12

If $d=1$, then for every $w\leq_T\gamma$ there exists a unique rising reduced T-decomposition of w.

Proof. The complex reflection group $G(1,1,n)$ is isomorphic to the Coxeter group A_{n-1}, and under this isomorphism, γ corresponds to the long cycle $(1\;2\;\ldots\;n)$. Then, $T_\gamma=\Big\{\big(\big(a^{(0)}\;b^{(0)}\big)\big)\mid 1\leq a<b\leq n\Big\}$ and $\prec_\gamma$ is the lexicographic order on T_γ, and the claim follows from [6, Theorem 3.5(ii)]. $\square$

The next lemma states what the coatoms of $\mathcal{NC}_{G(d,d,n)}$ look like.

Lemma 4.4.13

Let $t\in T_\gamma$. If $t=\big(\big(a^{(0)}\;b^{(0)}\big)\big)$, where $1\leq a<b<n$, then
$$\gamma t=\big[1^{(0)}\;\ldots\;a^{(0)}\;(b{+}1)^{(0)}\;\ldots\;(n{-}1)^{(0)}\big]\big[n^{(0)}\big]_{d-1}\big(\big(a{+}1^{(0)}\;\ldots\;b^{(0)}\big)\big).$$
If $t=\big(\big(a^{(0)}\;b^{(d-1)}\big)\big)$, where $1\leq a<b<n$, then
$$\gamma t=\big(\big(1^{(0)}\;\ldots\;a^{(0)}\;(b{+}1)^{(d-1)}\;\ldots\;(n{-}1)^{(d-1)}\big)\big)\big[a{+}1^{(0)}\;\ldots\;b^{(0)}\big]\big[n^{(0)}\big]_{d-1}.$$
If $t=\big(\big(a^{(0)}\;n^{(s)}\big)\big)$, where $1\leq a<n$ and $0\leq s<d$, then
$$\gamma t=\big(\big(1^{(0)}\;\ldots\;a^{(0)}\;n^{(s-1)}\;(a{+}1)^{(d-1)}\;\ldots\;(n{-}1)^{(d-1)}\big)\big).$$

Proof. This is a straightforward computation. $\square$

If we combine the previous lemma with Proposition 4.3.4, then we obtain what the parabolic subgroups of $G(d,d,n)$ can possibly look like.

Lemma 4.4.14

Let W be a parabolic subgroup of $G(d,d,n)$. If W is irreducible, then W is either isomorphic to $G(1,1,n')$ or to $G(d,d,n')$ for some $n' \leq n$. If W is reducible, then W is isomorphic to a direct product of irreducible parabolic subgroups of $G(d,d,n)$.

Proof. In view of Corollary 4.3.3 every parabolic subgroup of $G(d,d,n)$ is well-generated. The claim follows from [35, Fact 1.7] and [35, Table 2]. □

From now on, let $\prec_\gamma$ denote the total order of T_γ as defined in (4.13). First we focus on the rank-2 intervals of $\mathcal{NC}_{G(d,d,n)}(\gamma)$.

Lemma 4.4.15

Let $w \leq_T \gamma$ with $\ell_T(w) = 2$. There exists a unique rising reduced T-decomposition of w with respect to the restriction of $\prec_\gamma$ to the reflections in $T_\gamma \cap [\varepsilon, w]$.

Proof. Let $w = t_1t_2$ for $t_1, t_2 \in T_\gamma$. If t_1 and t_2 commute, then $w = t_1t_2 = t_2t_1$ are the only possible reduced T-decompositions of w. Since $\prec_\gamma$ is a total order there is nothing to show. Suppose that t_1 and t_2 do not commute. With the help of Proposition 4.4.8 we can explicitly determine the possible forms of w. Analogously to the proof of Proposition 4.4.8, we investigate the fixed space of $w^{-1}\gamma$ to determine which of these possibilities can actually occur in $\mathcal{NC}_{G(d,d,n)}(\gamma)$. The details of this investigation can be found in Appendix C.1. We state here only the relevant cases.

(i) Let $t_1 = \big(\big(a^{(0)}\ b^{(0)}\big)\big), t_2 = \big(\big(b^{(0)}\ c^{(0)}\big)\big)$, where $1 \leq a < b < c < n$. We have $w = \big(\big(a^{(0)}\ b^{(0)}\ c^{(0)}\big)\big)$, and the reduced T-decompositions of w are

$$w = \big(\big(a^{(0)}\ b^{(0)}\big)\big)\big(\big(b^{(0)}\ c^{(0)}\big)\big) = \big(\big(b^{(0)}\ c^{(0)}\big)\big)\big(\big(a^{(0)}\ c^{(0)}\big)\big) = \big(\big(a^{(0)}\ c^{(0)}\big)\big)\big(\big(a^{(0)}\ b^{(0)}\big)\big).$$

According to (4.13) only $w = \big(\big(a^{(0)}\ b^{(0)}\big)\big)\big(\big(b^{(0)}\ c^{(0)}\big)\big)$ is increasing.

(ii) Let $t_1 = \big(\big(a^{(0)}\ b^{(0)}\big)\big), t_2 = \big(\big(b^{(0)}\ c^{(d-1)}\big)\big)$, where $1 \leq a < b < c < n$. We have $w = \big(\big(a^{(0)}\ b^{(0)}\ c^{(d-1)}\big)\big)$, and the reduced T-decompositions of w are

$$w = \big(\big(a^{(0)}\ b^{(0)}\big)\big)\big(\big(b^{(0)}\ c^{(d-1)}\big)\big) = \big(\big(b^{(0)}\ c^{(d-1)}\big)\big)\big(\big(a^{(0)}\ c^{(d-1)}\big)\big) = \big(\big(a^{(0)}\ c^{(d-1)}\big)\big)\big(\big(a^{(0)}\ b^{(0)}\big)\big).$$

According to (4.13) only $w = \big(\big(a^{(0)}\ b^{(0)}\big)\big)\big(\big(b^{(0)}\ c^{(d-1)}\big)\big)$ is increasing.

(iii) Let $t_1 = \big(\big(a^{(0)}\ b^{(0)}\big)\big), t_2 = \big(\big(b^{(0)}\ n^{(s)}\big)\big)$, where $1 \leq a < b < n$ and $0 \leq s < d$. We have $w = \big(\big(a^{(0)}\ b^{(0)}\ n^{(s)}\big)\big)$, and the reduced T-decompositions of w are

$$w = \big(\big(a^{(0)}\ b^{(0)}\big)\big)\big(\big(b^{(0)}\ n^{(s)}\big)\big) = \big(\big(b^{(0)}\ n^{(s)}\big)\big)\big(\big(a^{(0)}\ n^{(s)}\big)\big) = \big(\big(a^{(0)}\ n^{(s)}\big)\big)\big(\big(a^{(0)}\ b^{(0)}\big)\big).$$

According to (4.13) only $w = \big(\big(a^{(0)}\ b^{(0)}\big)\big)\big(\big(b^{(0)}\ n^{(s)}\big)\big)$ is increasing.

(iv) Let $t_1 = \big(\big(b^{(0)}\ c^{(0)}\big)\big), t_2 = \big(\big(a^{(0)}\ c^{(d-1)}\big)\big)$, where $1 \leq a < b < c < n$. We have $w = \big(\big(a^{(0)}\ b^{(d-1)}\ c^{(d-1)}\big)\big)$, and the reduced T-decompositions w are

$$w = \big(\big(a^{(0)}\ b^{(d-1)}\big)\big)\big(\big(b^{(0)}\ c^{(0)}\big)\big) = \big(\big(b^{(0)}\ c^{(0)}\big)\big)\big(\big(a^{(0)}\ c^{(d-1)}\big)\big) = \big(\big(a^{(0)}\ c^{(d-1)}\big)\big)\big(\big(a^{(0)}\ b^{(d-1)}\big)\big).$$

According to (4.13) only $w = \big(\big(b^{(0)}\ c^{(0)}\big)\big)\big(\big(a^{(0)}\ c^{(d-1)}\big)\big)$ is increasing.

(v) Let $t_1 = \big(\!\big(a^{(0)}\ b^{(d-1)}\big)\!\big), t_2 = \big(\!\big(a^{(0)}\ n^{(s)}\big)\!\big)$, where $1 \leq a < b < n$ and $0 \leq s < d$. We have $w = \big(\!\big(a^{(0)}\ n^{(s)}\ b^{(d-1)}\big)\!\big)$, and the reduced T-decompositions of w are

$$w = \big(\!\big(a^{(0)}\ n^{(s)}\big)\!\big)\big(\!\big(b^{(0)}\ n^{(s+1)}\big)\!\big) = \big(\!\big(b^{(0)}\ n^{(s+1)}\big)\!\big)\big(\!\big(a^{(0)}\ b^{(d-1)}\big)\!\big) = \big(\!\big(a^{(0)}\ b^{(d-1)}\big)\!\big)\big(\!\big(a^{(0)}\ n^{(s)}\big)\!\big).$$

According to (4.13) only $w = \big(\!\big(a^{(0)}\ n^{(s)}\big)\!\big)\big(\!\big(b^{(0)}\ n^{(s+1)}\big)\!\big)$ is increasing.

(vi) Let $t_1 = \big(\!\big(a^{(0)}\ n^{(s)}\big)\!\big), t_2 = \big(\!\big(a^{(0)}\ n^{(t)}\big)\!\big)$, where $1 \leq a < n$ and $0 \leq s,t < d$ with $t \neq s$. We have $w = \big[a^{(0)}\big]_{t-s}\big[n^{(0)}\big]_{t-s}^{-1}$, and the reduced T-decompositions of w are

$$\begin{aligned} w &= \big(\!\big(a^{(0)}\ n^{(s)}\big)\!\big)\big(\!\big(a^{(0)}\ n^{(s+1)}\big)\!\big) = \big(\!\big(a^{(0)}\ n^{(s+1)}\big)\!\big)\big(\!\big(a^{(0)}\ n^{(s+2)}\big)\!\big) \\ &= \big(\!\big(a^{(0)}\ n^{(s+2)}\big)\!\big)\big(\!\big(a^{(0)}\ n^{(s+3)}\big)\!\big) = \cdots = \big(\!\big(a^{(0)}\ n^{(s-1)}\big)\!\big)\big(\!\big(a^{(0)}\ n^{(s)}\big)\!\big). \end{aligned}$$

According to (4.13) only $w = \big(\!\big(a^{(0)}\ n^{(0)}\big)\!\big)\big(\!\big(a^{(0)}\ n^{(1)}\big)\!\big)$ is increasing. $\square$

Now we consider the intervals of $\mathcal{NC}_{G(d,d,n)}(\gamma)$ that are isomorphic to $\mathcal{NC}_{G(1,1,n')}$ for some $n' \leq n$.

Proposition 4.4.16

Let $w \leq_T \gamma$ such that the parabolic subgroup of $G(d,d,n)$, in which w is a Coxeter element, is isomorphic to $G(1,1,n')$ for some $n' \leq n$. Then, w is of one of the following three forms:
(i) $w = \big(\!\big((a{+}1)^{(0)}\ (a{+}2)^{(0)}\ \ldots\ b^{(0)}\big)\!\big)$, where $1 \leq a < b < n$,
(ii) $w = \big(\!\big(1^{(0)}\ 2^{(0)}\ \ldots\ a^{(0)}\ (b{+}1)^{(d-1)}\ (b{+}2)^{(d-1)}\ \ldots\ (n{-}1)^{(d-1)}\big)\!\big)$, where $1 \leq a < b < n$, or
(iii) $w = \big(\!\big(1^{(0)}\ 2^{(0)}\ \ldots\ a^{(0)}\ n^{(s-1)}(a{+}1)^{(d-1)}\ (a{+}2)^{(d-1)}\ \ldots\ (n{-}1)^{(d-1)}\big)\!\big)$, where $1 \leq a < n$.
Moreover, in each of these cases there exists a unique rising reduced T-decomposition of w with respect to the restriction of $\prec_\gamma$ to the reflections in $T_\gamma \cap [\varepsilon, w]$.

Proof. The observation that w can only be of the forms (i)–(iii) is a straightforward computation using Proposition 4.4.8. The proof of the second part of this proposition is rather technical, and hence omitted here. The details can be found in Appendix C.2. We only present the unique rising reduced T-decompositions of w for the different cases:

(i) Let $w = \big(\!\big((a{+}1)^{(0)}\ (a{+}2)^{(0)}\ \ldots\ b^{(0)}\big)\!\big)$, where $1 \leq a < b < n$. The unique rising reduced T-decomposition of w is

$$w = \big(\!\big((a{+}1)^{(0)}\ (a{+}2)^{(0)}\big)\!\big)\big(\!\big((a{+}2)^{(0)}\ (a{+}3)^{(0)}\big)\!\big)\cdots\big(\!\big((b{-}1)^{(0)}\ b^{(0)}\big)\!\big).$$

(ii) Let $w = \big(\!\big(1^{(0)}\ 2^{(0)}\ \ldots\ a^{(0)}\ (b{+}1)^{(d-1)}\ (b{+}2)^{(d-1)}\ \ldots\ (n{-}1)^{(d-1)}\big)\!\big)$, where $1 \leq a < b < n$. The unique rising reduced T-decomposition of w is

$$\begin{aligned} w = \big(\!\big(1^{(0)}\ 2^{(0)}\big)\!\big)\big(\!\big(2^{(0)}\ 3^{(0)}\big)\!\big)\cdots\big(\!\big((a{-}1)^{(0)}\ a^{(0)}\big)\!\big) \\ \big(\!\big((b{+}1)^{(0)}\ (b{+}2)^{(0)}\big)\!\big)\cdots\big(\!\big((n{-}2)^{(0)}\ (n{-}1)^{(0)}\big)\!\big)\big(\!\big(a^{(0)}\ (n{-}1)^{(d-1)}\big)\!\big). \end{aligned}$$

(iii) Let $w = \big(\!\big(1^{(0)}\ 2^{(0)}\ \ldots\ a^{(0)}\ n^{(s-1)}(a{+}1)^{(d-1)}\ (a{+}2)^{(d-1)}\ \ldots\ (n{-}1)^{(d-1)}\big)\!\big)$, where $1 \leq a < n$. The unique rising reduced T-decomposition of w is

$$\begin{aligned} w = \big(\!\big(1^{(0)}\ 2^{(0)}\big)\!\big)\big(\!\big(2^{(0)}\ 3^{(0)}\big)\!\big)\cdots\big(\!\big((a{-}1)^{(0)}\ a^{(0)}\big)\!\big)\big(\!\big((a{+}1)^{(0)}\ (a{+}2)^{(0)}\big)\!\big) \\ \big(\!\big((a{+}2)^{(0)}\ (a{+}3)^{(0)}\big)\!\big)\cdots\big(\!\big((n{-}2)^{(0)}\ (n{-}1)^{(0)}\big)\!\big)\big(\!\big(a^{(0)}\ n^{(s-1)}\big)\!\big)\big(\!\big((n{-}1)^{(0)}\ n^{(s)}\big)\!\big). \end{aligned}$$

$\square$

The following corollary is immediate.

Corollary 4.4.17

Let $w \leq_T \gamma$ such that the parabolic subgroup W of $G(d,d,n)$, in which w is a Coxeter element, is reducible, and hence $W = W_1 \times W_2 \times \cdots \times W_l$ for some l. If for each $i \in \{1,2,\ldots,l\}$, the group W_i is isomorphic to $G(1,1,n_i)$ for $n_i \leq n$, then there exists a unique rising reduced T-decomposition of w with respect to the restriction of $\prec_\gamma$ to the reflections in $T_\gamma \cap [\varepsilon, w]$.

Proof. This works analogously to the proof of Proposition 4.4.16. See Appendix C.3 for the details. □

Finally we focus on the intervals of $\mathcal{NC}_{G(d,d,n)}(\gamma)$ that are isomorphic to $\mathcal{NC}_{G(d,d,n')}$ for some $n' < n$.

Proposition 4.4.18

Let $w \leq_T \gamma$ such that the parabolic subgroup of $G(d,d,n)$, in which w is a Coxeter element, is isomorphic to $G(d,d,n')$ for some $n' < n$. There exists a unique rising reduced T-decomposition of w with respect to the restriction of $\prec_\gamma$ to the reflections in $T_\gamma \cap [\varepsilon, w]$.

Proof. Again we proceed by induction on $\ell_T(w)$, and the case $\ell_T(w) = 2$ is covered by Lemma 4.4.15. In view of Lemma 4.3.6, we can assume that $w = \gamma$, and that the claim is true for all $w' <_T w$ that satisfy the condition. We notice immediately that the simple decomposition of γ, namely

$$\gamma = s_1 s_2 \cdots s_n = \big(\big(1^{(0)}\;2^{(0)}\big)\big)\big(\big(2^{(0)}\;3^{(0)}\big)\big)\cdots\big(\big((n-1)^{(0)}\;n^{(0)}\big)\big)\big(\big((n-1)^{(0)}\;n^{(1)}\big)\big)$$

is rising with respect to (4.13). Let $\gamma = t_1 t_2 \cdots t_n$ be a rising reduced T-decomposition of γ that is different from $s_1 s_2 \cdots s_n$, and let k be the maximal index where $t_k \neq s_k$. If $k < n$, then $\gamma s_n s_{n-1} \cdots s_{k+1} = \big(\big(1^{(0)}\;2^{(0)}\;\ldots\;(k+1)^{(0)}\big)\big)$. It follows from Proposition 4.4.16 that the only rising reduced T-decomposition of $\gamma s_n s_{n-1} \cdots s_{k+1}$ is $s_1 s_2 \cdots s_k$, which is a contradiction. Hence let $k = n$. In view of Proposition 4.4.8, there are essentially three possible choices of t_n, and we write $\gamma' = \gamma t_n$. Moreover, let W denote the parabolic subgroup of $G(d,d,n)$ in which γ' is a Coxeter element.

(i) Let $t_n = \big(\big(a^{(0)}\;b^{(0)}\big)\big)$, where $1 \leq a < b < n$. Lemma 4.4.13 implies that we can write $\gamma' = w_1 w_2$ with

$$w_1 = \big[1^{(0)}\;2^{(0)}\;\ldots\;a^{(0)}\;(b+1)^{(0)}\;(b+2)^{(0)}\;\ldots\;(n-1)^{(0)}\big]\big[n^{(0)}\big]_{d-1}, \quad \text{and}$$
$$w_2 = \big(\big((a+1)^{(0)}\;(a+2)^{(0)}\;\ldots\;b^{(0)}\big)\big).$$

This implies that w_1 is a Coxeter element in a parabolic subgroup W_1 of $G(d,d,n)$ isomorphic to $G(d,d,n+a-b)$, and w_2 is a Coxeter element in a parabolic subgroup W_2 of $G(d,d,n)$ isomorphic to $G(1,1,b-a-1)$, and we can write $W = W_1 \times W_2$. By induction hypothesis and by Proposition 4.4.16 there exist unique rising reduced T-decompositions of w_1 and w_2, namely

$$\begin{aligned} w_1 = {} & \big(\big(1^{(0)}\;2^{(0)}\big)\big)\big(\big(2^{(0)}\;3^{(0)}\big)\big)\cdots\big(\big((a-1)^{(0)}\;a^{(0)}\big)\big)\big(\big(a^{(0)}\;(b+1)^{(0)}\big)\big)\\ & \big(\big((b+1)^{(0)}\;(b+2)^{(0)}\big)\big)\cdots\big(\big((n-1)^{(0)}\;n^{(0)}\big)\big)\big(\big((n-1)^{(0)}\;n^{(1)}\big)\big), \quad \text{and}\\ w_2 = {} & \big(\big((a+1)^{(0)}\;(a+2)^{(0)}\big)\big)\big(\big((a+2)^{(0)}\;(a+3)^{(0)}\big)\big)\cdots\big(\big((b-1)^{(0)}\;b^{(0)}\big)\big). \end{aligned}$$

It is immediate to see that

$$\gamma' = \big(\!\big(1^{(0)}\ 2^{(0)}\big)\!\big) \cdots \big(\!\big((a-1)^{(0)}\ a^{(0)}\big)\!\big) \big(\!\big(a^{(0)}\ (b+1)^{(0)}\big)\!\big) \big(\!\big((a+1)^{(0)}\ (a+2)^{(0)}\big)\!\big) \cdots \\ \big(\!\big((b-1)^{(0)}\ b^{(0)}\big)\!\big) \big(\!\big((b+1)^{(0)}\ (b+2)^{(0)}\big)\!\big) \cdots \big(\!\big((n-1)^{(0)}\ n^{(0)}\big)\!\big) \big(\!\big((n-1)^{(0)}\ n^{(1)}\big)\!\big)$$

is the unique rising reduced T-decomposition of γ' and hence has to correspond to $t_1 t_2 \cdots t_{n-1}$. However, we have for instance $\big(\!\big((n-1)^{(0)}\ n^{(1)}\big)\!\big) \succ_\gamma \big(\!\big(a^{(0)}\ b^{(0)}\big)\!\big) = t_n$, which contradicts the assumption that $t_1 t_2 \cdots t_n$ is rising.

(ii) Let $t_n = \big(\!\big(a^{(0)}\ b^{(d-1)}\big)\!\big)$, where $1 \leq a < b < n$. Lemma 4.4.13 implies that we can write $\gamma' = w_1 w_2$ with

$$w_1 = \big(\!\big(1^{(0)}\ 2^{(0)}\ \ldots\ a^{(0)}\ (b+1)^{(d-1)}\ (b+2)^{(d-1)}\ \ldots\ (n-1)^{(d-1)}\big)\!\big), \quad \text{and} \\ w_2 = \big[(a+1)^{(0)}\ (a+2)^{(0)}\ \ldots\ b^{(0)}\big] \big[n^{(0)}\big]_{d-1}.$$

This implies that w_1 is a Coxeter element in a parabolic subgroup W_1 of $G(d,d,n)$ isomorphic to $G(1,1,n+a-b-2)$, and w_2 is a Coxeter element in a parabolic subgroup W_2 of $G(d,d,n)$ isomorphic to $G(d,d,b-a+1)$, and we can write $W = W_1 \times W_2$. By induction hypothesis and by Proposition 4.4.16 there exist unique rising reduced T-decompositions of w_1 and w_2, namely

$$w_1 = \big(\!\big(1^{(0)}\ 2^{(0)}\big)\!\big) \big(\!\big(2^{(0)}\ 3^{(0)}\big)\!\big) \cdots \big(\!\big((a-1)^{(0)}\ a^{(0)}\big)\!\big) \\ \big(\!\big((b+1)^{(0)}\ (b+2)^{(0)}\big)\!\big) \cdots \big(\!\big((n-2)^{(0)}\ (n-1)^{(0)}\big)\!\big) \big(\!\big(a^{(0)}\ (n-1)^{(d-1)}\big)\!\big), \quad \text{and} \\ w_2 = \big(\!\big((a+1)^{(0)}\ (a+2)^{(0)}\big)\!\big) \big(\!\big((a+2)^{(0)}\ (a+3)^{(0)}\big)\!\big) \cdots \big(\!\big((b-1)^{(0)}\ b^{(0)}\big)\!\big) \big(\!\big(b^{(0)}\ n^{(0)}\big)\!\big) \big(\!\big(b^{(0)}\ n^{(1)}\big)\!\big).$$

It is immediate to see that

$$\gamma' = \big(\!\big(1^{(0)}\ 2^{(0)}\big)\!\big) \cdots \big(\!\big((a-1)^{(0)}\ a^{(0)}\big)\!\big) \big(\!\big((a+1)^{(0)}\ (a+2)^{(0)}\big)\!\big) \cdots \big(\!\big((b-1)^{(0)}\ b^{(0)}\big)\!\big) \\ \big(\!\big((b+1)^{(0)}\ (b+2)^{(0)}\big)\!\big) \cdots \big(\!\big((n-2)^{(0)}\ (n-1)^{(0)}\big)\!\big) \big(\!\big(a^{(0)}\ (n-1)^{(d-1)}\big)\!\big) \big(\!\big(b^{(0)}\ n^{(0)}\big)\!\big) \big(\!\big(b^{(0)}\ n^{(1)}\big)\!\big)$$

is the unique rising reduced T-decomposition of γ' and hence has to correspond to $t_1 t_2 \cdots t_{n-1}$. However, we have for instance $\big(\!\big(b^{(0)}\ n^{(1)}\big)\!\big) \succ_\gamma \big(\!\big(a^{(0)}\ b^{(d-1)}\big)\!\big) = t_n$, which contradicts the assumption that $t_1 t_2 \cdots t_n$ is rising.

(iii) Let $t = \big(\!\big(a^{(0)}\ n^{(s)}\big)\!\big)$, where $1 \leq a < n-1$ and $0 \leq s < d$. Lemma 4.4.13 implies that we can write

$$\gamma' = \big(\!\big(1^{(0)}\ 2^{(0)}\ \ldots\ a^{(0)}\ n^{(s-1)} (a+1)^{(d-1)}\ (a+2)^{(d-1)}\ \ldots\ (n-1)^{(d-1)}\big)\!\big).$$

In view of Proposition 4.4.16 there exists a unique rising reduced T-decomposition of γ, namely

$$\gamma' = \big(\!\big(1^{(0)}\ 2^{(0)}\big)\!\big) \cdots \big(\!\big((a-1)^{(0)}\ a^{(0)}\big)\!\big) \big(\!\big((a+1)^{(0)}\ (a+2)^{(0)}\big)\!\big) \cdots \big(\!\big((n-2)^{(0)}\ (n-1)^{(0)}\big)\!\big) \\ \big(\!\big(a^{(0)}\ n^{(s-1)}\big)\!\big) \big(\!\big((n-1)^{(0)}\ n^{(s)}\big)\!\big),$$

and this decomposition has to correspond to $t_1 t_2 \cdots t_{n-1}$. However, we have for instance $\big(\!\big((n-1)^{(0)}\ n^{(s)}\big)\!\big) \succ_\gamma \big(\!\big(a^{(0)}\ n^{(s)}\big)\!\big) = t_n$, which contradicts the assumption that $t_1 t_2 \cdots t_n$ is rising.

(iv) Let $t = \big(\!\big((n-1)^{(0)}\ n^{(s)}\big)\!\big)$, where $0 \leq s < d$. It follows that $s \neq 1$, because otherwise $t_n = s_n$. Lemma 4.4.13 implies that we can write

$$\gamma' = \big(\!\big(1^{(0)}\ 2^{(0)}\ \ldots\ (n-1)^{(0)}\ n^{(s-1)}\big)\!\big).$$

In view of Proposition 4.4.16 there exists a unique rising reduced T-decomposition of γ, namely

$$\gamma' = \big(\!\big(1^{(0)}\ 2^{(0)}\big)\!\big) \cdots \big(\!\big((n-2)^{(0)}\ (n-1)^{(0)}\big)\!\big)\big(\!\big((n-1)^{(0)}\ n^{(s-1)}\big)\!\big),$$

and this decomposition has to correspond to $t_1t_2\cdots t_{n-1}$. However, since $s\neq 1$, we have for instance $\big(\!\big((n-1)^{(0)}\ n^{(s-1)}\big)\!\big) \succ_\gamma \big(\!\big((n-1)^{(0)}\ n^{(s)}\big)\!\big) = t_n$, which contradicts the assumption that $t_1t_2\cdots t_n$ is rising.

Hence $\gamma = s_1s_2\cdots s_n$ is the unique rising reduced T-decomposition of γ. □

The following corollary is immediate.

Corollary 4.4.19

Let $w \leq_T \gamma$ such that the parabolic subgroup W of $G(d,d,n)$, in which w is a Coxeter element, is reducible. There exists a unique rising reduced T-decomposition of w with respect to the restriction of $\prec_\gamma$ to the reflections in $T_\gamma \cap [\varepsilon, w]$.

Proof. Since W is reducible, we can write $W = W_1 \times W_2 \times \cdots \times W_l$ for some l. It follows for instance from [35, Fact 1.7] and [35, Table 2] that at most one W_i is isomorphic to $G(d,d,n')$ for some $n' < n$, and the other W_j are isomorphic to $G(1,1,n_j)$ for $n_j \leq n$. The proof works analogously to the proofs of Corollary 4.4.17 and Proposition 4.4.18. □

Now we have collected all the ingredients for the proof of Theorem 4.4.11.

Proof of Theorem 4.4.11. We need to show that under the given assumptions in every interval $[u,v]$ of $\mathcal{NC}_{G(d,d,n)}(\gamma)$ there exists a unique rising maximal chain, and this maximal chain is lexicographically first. In view of Lemma 4.3.6, it suffices to consider intervals of the form $[\varepsilon, w]$. Proposition 4.4.5 implies that all reflections of $G(d,d,n)$ have order two. Hence we can apply Proposition 4.3.9, and we obtain that the lexicographically first maximal chain in $[\varepsilon, w]$ is rising. Now, Propositions 4.4.16 and 4.4.18 as well as Corollaries 4.4.17 and 4.4.19 imply together with Lemma 4.3.5 that there is exactly one rising maximal chain in $[\varepsilon, w]$, and we are done. □

Example 4.4.20

The lattice $\mathcal{NC}_{G(5,5,3)}(\gamma)$ with the edge-labeling λ_γ is depicted in Figure 43. The labels are explained in Example 4.4.9. Indeed, this labeling is an EL-labeling, and the unique rising maximal chain from ε to γ is indicated by thick edges.

4.4.2. The Exceptional Groups. In this section we prove Theorem 4.4.1 for the exceptional well-generated complex reflection groups and $m = 1$ by computing a total order on T_γ that makes the labeling λ_γ an EL-labeling of the corresponding lattice of noncrossing partitions. This computation is done by a computer program, called Lins [81]. Given a well-generated complex reflection group W, Lins computes some Coxeter element $\gamma \in W$ and starts with an arbitrary total order on T_γ. It successively adapts this total order until for each rank-2 interval only one rising maximal chain remains. In the end it checks that this total order indeed makes λ_γ an EL-labeling for the whole lattice $\mathcal{NC}_W(\gamma)$. (The observation that this process always works led us to Conjecture 4.4.28 in Section 4.4.5.) This algorithm, however, is not deterministic meaning that different runs of Lins may produce different total orders. It uses Michel's GAP-distribution [80] and Borchmann's FCA-tool [28] for computing the chains of the lattice. For more information on Formal Concept Analysis (FCA), see the monograph

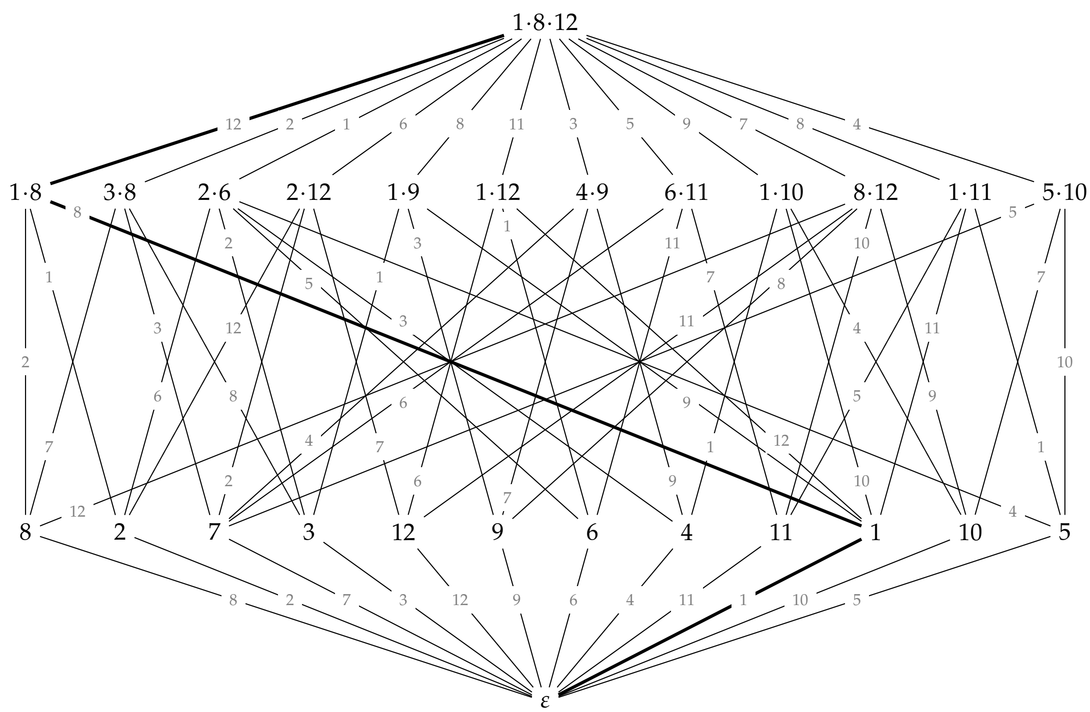


FIGURE 43. The lattice $\mathcal{NC}_{G(5,5,3)}(\gamma)$ with the edge-labeling from (4.2). The labels are explained in Example 4.4.9.

[57]. LINS outputs several files, including some GAP scripts, a file containing the labeled chains as well as a file containing the total order on T_γ. For that it names the reflections of W abstractly by s_k, where $k \in \{1,2,\ldots,|NC_W(\gamma)|\}$. The index k comes from the internal representation of the group elements of W in GAP. This naming is deterministic, and it can be resolved with the GAP script used by LINS, which can separately be downloaded from `http://homepage.univie.ac.at/henri.muehle/files/lins`. The following theorem is the main result of this section.

THEOREM 4.4.21

The lattice $\mathcal{NC}_W$ is EL-shellable for every exceptional well-generated complex reflection group.

PROOF. We distinguish different classes of exceptional well-generated complex reflection groups.

(i) The groups $G_{23}, G_{28}, G_{30}, G_{35}, G_{36}, G_{37}$. These are the exceptional complex reflection groups that are isomorphic to the exceptional Coxeter groups, see [35, p. 6]. Hence the claim follows from [6].

(ii) The groups G_{25}, G_{26}, G_{32}. By Proposition 4.4.2, the lattices of noncrossing partitions associated with these groups are isomorphic to the lattices of noncrossing partitions of the Coxeter groups A_3, B_3 and A_4, respectively. Hence the claim follows again from [6].

(iii) The groups $G_4, G_5, G_6, G_8, G_9, G_{10}, G_{14}, G_{16}, G_{17}, G_{18}, G_{20}, G_{21}$. These groups are the exceptional well-generated complex reflection groups of rank 2. Hence their associated lattices of noncrossing partitions have rank 2 as well, and thus each such lattice is isomorphic to some

Group	Total order
G_{24}	$s_{26} \prec s_5 \prec s_3 \prec s_{29} \prec s_{21} \prec s_{28} \prec s_{18} \prec s_7 \prec s_2 \prec s_4 \prec s_{11} \prec s_8 \prec s_{23} \prec s_{25}$
G_{27}	$s_{23} \prec s_{38} \prec s_{42} \prec s_{15} \prec s_{36} \prec s_{29} \prec s_{33} \prec s_{27} \prec s_{18} \prec s_{13} \prec s_4 \prec s_3 \prec s_2 \prec s_8 \prec s_5 \prec s_{21} \prec s_{17} \prec s_{34} \prec s_{37} \prec s_{30}$
G_{29}	$s_{101} \prec s_4 \prec s_{76} \prec s_{109} \prec s_8 \prec s_{105} \prec s_{64} \prec s_{47} \prec s_6 \prec s_{33} \prec s_{68} \prec s_{13} \prec s_{20} \prec s_{39} \prec s_{93} \prec s_9 \prec s_{88} \prec s_2 \prec s_{70} \prec s_{28} \prec s_{110} \prec s_{25} \prec s_{53} \prec s_3 \prec s_{18}$
G_{33}	$s_5 \prec s_{13} \prec s_7 \prec s_{33} \prec s_{56} \prec s_{19} \prec s_{36} \prec s_{58} \prec s_{47} \prec s_{182} \prec s_{16} \prec s_{17} \prec s_{224} \prec s_{281} \prec s_{297} \prec s_{42} \prec s_{179} \prec s_{217} \prec s_{89} \prec s_{128} \prec s_{86} \prec s_{110} \prec s_2 \prec s_{172} \prec s_{277} \prec s_{169} \prec s_{76} \prec s_{68} \prec s_3 \prec s_{12}$
G_{34}	$s_{1568} \prec s_{937} \prec s_{1361} \prec s_{213} \prec s_{13} \prec s_{142} \prec s_{669} \prec s_{888} \prec s_{58} \prec s_7 \prec s_{65} \prec s_{67} \prec s_{480} \prec s_{295} \prec s_8 \prec s_{37} \prec s_{40} \prec s_{256} \prec s_{714} \prec s_{1060} \prec s_{1447} \prec s_{17} \prec s_3 \prec s_{117} \prec s_{53} \prec s_{1252} \prec s_{639} \prec s_{62} \prec s_6 \prec s_{702} \prec s_{915} \prec s_{1043} \prec s_{43} \prec s_{359} \prec s_{428} \prec s_{23} \prec s_4 \prec s_{75} \prec s_{127} \prec s_{191} \prec s_{368} \prec s_{157} \prec s_{648} \prec s_{1234} \prec s_{181} \prec s_2 \prec s_{683} \prec s_{49} \prec s_{264} \prec s_{235} \prec s_{905} \prec s_{1241} \prec s_{60} \prec s_{1558} \prec s_{1353} \prec s_{319}$

FIGURE 44. Explicit total orders of the reflections in $\mathcal{NC}_W(\gamma)$, where $W \in \{G_{24}, G_{27}, G_{29}, G_{33}, G_{34}\}$, that make λ_γ an EL-labeling.

lattice of noncrossing partitions of some Coxeter group $I_2(d)$ for some d. Once more, the claim follows from [6].

(iv) The groups $G_{24}, G_{27}, G_{29}, G_{33}, G_{34}$. These groups are the remaining groups whose lattices of noncrossing partitions are not related to any previously known case. Figure 44 provides total orders of the reflections of $\mathcal{NC}_W(\gamma)$, where W is one of these groups, and where γ is a particular choice Coxeter element. These orders were computed with LINS. $\square$

4.4.3. The Case $m > 1$. In this section we prove the general case of Theorem 4.4.1, namely where $m > 1$. We basically apply the same construction that ARMSTRONG and THOMAS have applied in their proof of [1, Theorem 3.7.2], which is Theorem 4.4.1 restricted to finite Coxeter groups. This construction uses several structural properties of $\mathcal{NC}_W^{(m)}$ that were presented in [1] provided that W is a Coxeter group. However, all of these properties generalize straightforwardly to well-generated complex reflection groups. Thus we can now proof Theorem 4.4.1.

PROOF OF THEOREM 4.4.1. First suppose that $m = 1$. In this case the claim follows from [6, Theorem 1.1] and from Proposition 4.4.2 and Theorems 4.4.11 and 4.4.21.

Now let $m > 1$, let W be a well-generated complex reflection group and let $\gamma \in W$ be a fixed Coxeter element. Moreover, let $\prec_\gamma$ be a total order of T_γ such that λ_γ is an EL-labeling of $\mathcal{NC}_W(\gamma)$. In particular, let us write $T_\gamma = \{t_1, t_2, \ldots, t_N\}$ where the elements are ordered increasingly with respect to $\prec_\gamma$.

We first define an EL-labeling of $\mathcal{NC}_{W^m}$, where W^m denotes the m-fold direct product of W with itself. For $i \in \{1,2,\ldots,m\}$ and $j \in \{1,2,\ldots,N\}$, define a vector $\mathbf{t}_{i,j} = (\varepsilon, \varepsilon, \ldots, \varepsilon, t_j, \varepsilon, \varepsilon, \ldots, \varepsilon)^{\mathsf{T}}$, where $t_j \in T_\gamma$ appears in the i-th coordinate. Define the set $T_\gamma^m = \{\mathbf{t}_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq N\}$, and consider the edge-labeling

$$\begin{aligned}\lambda_\gamma^m : \mathcal{E}\big(\mathcal{NC}_{W^m}(\gamma)\big) &\to T_\gamma^m,\\ \big((u_1, u_2, \ldots, u_m), (v_1, v_2, \ldots, v_m)\big) &\mapsto \big(\lambda_\gamma(u_1, v_1), \lambda_\gamma(u_2, v_2), \ldots, \lambda_\gamma(u_m, v_m)\big),\end{aligned}$$

where we use the additional convention that $\lambda_\gamma(w,w) = \varepsilon$ for all $W \in NC_W(\gamma)$. If we consider the following total order $\prec_\gamma^m$ of T_γ^m:

$$\mathbf{t}_{1,1} \prec_\gamma^m \mathbf{t}_{1,2} \prec_\gamma^m \cdots \prec_\gamma^m \mathbf{t}_{1,N} \prec_\gamma^m \mathbf{t}_{2,1} \prec_\gamma^m \mathbf{t}_{2,2} \prec_\gamma^m \cdots \prec_\gamma^m \mathbf{t}_{2,N} \prec_\gamma^m \mathbf{t}_{3,1} \prec_\gamma^m \cdots \prec_\gamma^m \mathbf{t}_{m,N},$$

then [21, Theorem 4.3] implies that λ_γ^m is an EL-labeling of $\mathcal{NC}_{W^m}(\gamma)$. A straightforward generalization of [1, Lemma 3.4.3] implies that $\mathcal{NC}_W^{(m)}(\gamma)$ is an up-set in the dual of $\mathcal{NC}_{W^m}(\gamma)$, i.e. it is a subposet of the dual of $\mathcal{NC}_{W^m}(\gamma)$ that is closed under taking upper covers. It follows immediately that λ_γ^m restricts to an edge-labeling of $\mathcal{NC}_W^{(m)}(\gamma)$ by reversing the order $\prec_\gamma^m$. Recall that $\overline{\mathcal{NC}}_W^{(m)}(\gamma)$ is the lattice that is constructed from $\mathcal{NC}_W^{(m)}(\gamma)$ by adding a least element $\mathbf{0}$. Armstrong and Thomas introduce an abstract symbol δ, and define an edge-labeling $\lambda_\gamma^{(m)}$ of $\overline{\mathcal{NC}}_W^{(m)}(\gamma)$ as follows: let $T_\gamma^{(m)} = T_\gamma^m \cup \{\delta\}$ and define

$$\lambda_\gamma^{(m)} : \mathcal{E}\big(\overline{\mathcal{NC}}_W^{(m)}(\gamma)\big) \to T_\gamma^{(m)}, \quad (\mathbf{u}, \mathbf{v}) \mapsto \begin{cases} \delta, & \text{if } \mathbf{u} = \mathbf{0},\\ \lambda_\gamma^m(\mathbf{v}, \mathbf{u}), & \text{otherwise.}\end{cases}$$

Consider the following total order $\prec_\gamma^{(m)}$ on $T_\gamma^{(m)}$:

$$\begin{aligned}\mathbf{t}_{m,N} \prec_\gamma^{(m)} \mathbf{t}_{m,N-1} \prec_\gamma^{(m)} \cdots \prec_\gamma^{(m)} \mathbf{t}_{m,1} \prec_\gamma^{(m)} \mathbf{t}_{m-1,N}&\\ \prec_\gamma^{(m)} \mathbf{t}_{m-1,N-1} \prec_\gamma^{(m)} \cdots \prec_\gamma^{(m)} \mathbf{t}_{2,1} \prec_\gamma^{(m)} \delta \prec_\gamma^{(m)} \mathbf{t}_{1,N} \prec_\gamma^{(m)} \mathbf{t}_{1,N-1} \prec_\gamma^{(m)} \cdots \prec_\gamma^{(m)} \mathbf{t}_{1,1}.&\end{aligned} \tag{4.15}$$

The proof that $\lambda_\gamma^{(m)}$ is an EL-labeling of $\overline{\mathcal{NC}}_W^{(m)}(\gamma)$, when $T_\gamma^{(m)}$ is ordered by $\prec_\gamma^{(m)}$ works now verbatim to the proof of [1, Theorem 3.7.2]. $\square$

EXAMPLE 4.4.22

Let us again consider the group $G(5,5,3)$, let γ be the Coxeter element of $G(5,5,3)$ as defined in (4.7), and identify the reflections in T_γ by their position in (4.14). For $m = 2$ the total order of $T_\gamma^{(2)}$ according to (4.15) is:

$$\begin{aligned}(\varepsilon,12) &\prec_\gamma^{(2)} (\varepsilon,11) \prec_\gamma^{(2)} (\varepsilon,10) \prec_\gamma^{(2)} (\varepsilon,9) \prec_\gamma^{(2)} (\varepsilon,8) \prec_\gamma^{(2)} (\varepsilon,7)\\ &\prec_\gamma^{(2)} (\varepsilon,6) \prec_\gamma^{(2)} (\varepsilon,5) \prec_\gamma^{(2)} (\varepsilon,4) \prec_\gamma^{(2)} (\varepsilon,3) \prec_\gamma^{(2)} (\varepsilon,2)\\ &\prec_\gamma^{(2)} (\varepsilon,1) \prec_\gamma^{(2)} \delta \prec_\gamma^{(2)} (12,\varepsilon) \prec_\gamma^{(2)} (11,\varepsilon) \prec_\gamma^{(2)} (10,\varepsilon)\end{aligned}$$

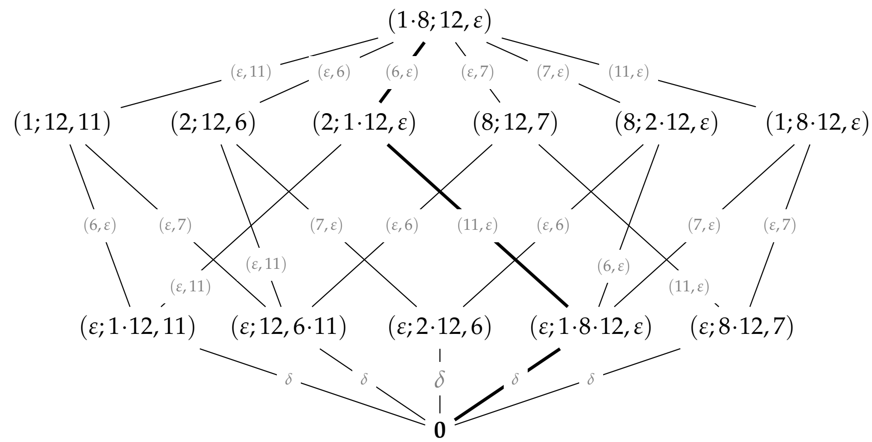


FIGURE 45. An interval in $\overline{\mathcal{NC}}^{(2)}_{G(5,5,3)}(\gamma)$ with the labeling $\lambda^{(2)}_{\gamma}$. The labels are explained in Example 4.4.9.

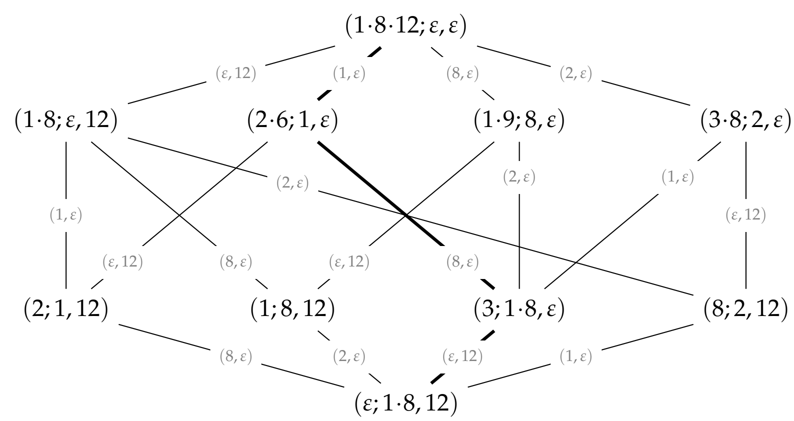


FIGURE 46. Another interval in $\overline{\mathcal{NC}}^{(2)}_{G(5,5,3)}(\gamma)$ with the labeling $\lambda^{(2)}_{\gamma}$. The labels are explained in Example 4.4.9.

$$\prec^{(2)}_{\gamma} (9,\varepsilon) \quad \prec^{(2)}_{\gamma} (8,\varepsilon) \quad \prec^{(2)}_{\gamma} (7,\varepsilon) \quad \prec^{(2)}_{\gamma} (6,\varepsilon) \quad \prec^{(2)}_{\gamma} (5,\varepsilon)$$
$$\prec^{(2)}_{\gamma} (4,\varepsilon) \quad \prec^{(2)}_{\gamma} (3,\varepsilon) \quad \prec^{(2)}_{\gamma} (2,\varepsilon) \quad \prec^{(2)}_{\gamma} (1,\varepsilon).$$

The tuple $(\varepsilon, 8)$, for instance, represents the tuple $\Big(\varepsilon, \big((2^{(0)}\ 3^{(0)})\big)\Big)$. Figures 45 and 46 display two intervals of $\overline{\mathcal{NC}}^{(2)}_{G(5,5,3)}(\gamma)$ with the EL-labeling $\lambda^{(2)}_{\gamma}$. The nodes in each of these lattices are labeled by tuples that correspond to 2-divisible noncrossing partitions of $G(5,5,3)$ analogously to the labeling of the nodes of $\mathcal{NC}_{G(5,5,3)}(\gamma)$ in Figure 43. In each figure the unique rising maximal chain is indicated by thick edges.

4.4.4. The Möbius Function. In this section, we use Theorem 4.4.1 to compute the Möbius invariant of a certain subposet of $\overline{\mathcal{NC}}^{(m)}_{W}$, where W is a well-generated complex reflection group. More precisely, we obtain the following result.

PROPOSITION 4.4.23

Let W be a well-generated complex reflection group, let $\gamma \in W$ be a Coxeter element, and let $m > 0$. Further, let $\widehat{\mathcal{NC}_W^{(m)}}(\gamma)$ denote the lattice that arises from $\mathcal{NC}_W^{(m)}(\gamma)$ by removing the minimal elements and adding a least element instead. We have

$$\mu\Big(\widehat{\mathcal{NC}_W^{(m)}}(\gamma)\Big) = Cat^{(-m-1)}(W) - Cat^{(-m)}(W).$$

PROOF. This result was proven in [123] in the case that W is a Coxeter group by using an EL-labeling of $\overline{\mathcal{NC}}_W^{(m)}(\gamma)$. In view of Theorem 4.4.1 we can find an EL-labeling of $\overline{\mathcal{NC}}_W^{(m)}(\gamma)$ for every well-generated complex reflection group W, and thus we can carry over the proof of [123, Theorem 1.1] essentially verbatim. $\square$

We immediately obtain the following corollary, which was first observed for Coxeter groups in [2, Theorem 3]. The enumerative part for well-generated complex reflection groups is [2, Theorem 9].

COROLLARY 4.4.24

Let W be a well-generated complex reflection group of rank n, and let $m > 0$. The order complex of the poset $\mathcal{NC}_W^{(m)}$ with maximal and minimal elements removed is homotopy equivalent to a wedge of $(-1)^n\Big(Cat^{(-m-1)}(W) - Cat^{(-m)}(W)\Big)$-many $(n-2)$-spheres.

PROOF. By combining Proposition 1.1.14 with Theorems 1.1.16 and 1.1.5 we see that the order complex in question is homotopy equivalent to a wedge of $(n-2)$ spheres. The number of spheres involved is given by the Möbius invariant of the lattice $\widehat{\mathcal{NC}_W^{(m)}}(\gamma)$ from Proposition 4.4.23, and we are done. $\square$

EXAMPLE 4.4.25

Let us finish the running example of $G(5,5,3)$. For $m = 1$, we have

$$\mathrm{Cat}^{(-1)}(G(5,5,3)) = 0 \quad \text{and} \quad \mathrm{Cat}^{(-2)}(G(5,5,3)) = -17.$$

According to Corollary 4.4.24, the truncated order complex of $\mathcal{NC}_{G(5,5,3)}$ is homotopy equivalent to a wedge of 17 one-dimensional spheres, and it follows from Theorem 1.1.16 that there must be 17 falling maximal chains in $\mathcal{NC}_{G(5,5,3)}$. This can be checked easily by inspecting Figure 43. According to Proposition 4.4.23, the Möbius invariant of $\mathcal{NC}_{G(5,5,3)}$ is given by $\mu\big(\mathcal{NC}_{G(5,5,3)}\big) = -17$, which can again be checked by inspecting Figure 43.

4.4.5. Towards a Uniform Approach. In this section we sketch a possible uniform approach to Theorem 4.4.1. It suffices to have a uniform approach for the case $m = 1$ since the transition from $m = 1$ to $m > 1$ does not involve any case-by-case analysis. Moreover, in the case where W is a Coxeter group Theorem 1 in [6] provides a uniform proof of Theorem 4.4.1, while our generalization of their result relies on a case-by-case analysis of the well-generated complex reflection groups that are no Coxeter groups. However, we conjecture that the ideas used in [6] might indeed yield a uniform proof of Theorem 4.4.1, once they are properly generalized to the complex case. In [6, Definition 3.1] ATHANASIADIS, BRADY and WATT define a reflection ordering that is compatible with a chosen Coxeter element. Subsequently they show in [6, Theorem 3.5(ii)] that if T_γ is ordered by such a compatible reflection ordering, then λ_γ is an EL-labeling of $\mathcal{NC}_W(\gamma)$ provided that W is a Weyl group. Moreover, in [6, Section 4] they uniformly construct a compatible reflection ordering for all Coxeter groups, and they show that this yields an EL-labeling of the corresponding lattice of noncrossing partitions. In the next definition we generalize this concept.

DEFINITION 4.4.26

Let W be a well-generated complex reflection group, let $\gamma \in W$ be a Coxeter element. A total order $\prec$ of T_γ is *γ-compatible* if for every rank-2 interval of $\mathcal{NC}_W(\gamma)$ there exists a unique rising maximal chain with respect to the edge-labeling λ_γ defined in (4.2).

We have the following nice property.

PROPOSITION 4.4.27

Let W be a well-generated complex reflection group, and let $\gamma \in W$ be a Coxeter element. Then, there exists a γ-compatible reflection order of T_γ.

PROOF. Theorem 4.1 in [6] states (uniformly) that for every Coxeter group there exists Coxeter element γ such that we can find a γ-compatible reflection order (in the sense of [6]). Since these orders are also γ-compatible in our sense Proposition 4.2.4 implies that the claim holds for Coxeter groups.

Proposition 4.4.2 implies that $\mathcal{NC}_{G(d,1,n)} \cong \mathcal{NC}_{B_n}$, and in view of the previous paragraph, we conclude that the claim is true for the groups $G(d,1,n)$, where $d,n \geq 2$.

Now suppose that $W = G(d,d,n)$ for $d,n \geq 3$. Lemma 4.4.15 implies that the total order of T_γ defined in (4.13) is γ-compatible in our sense, where γ is the Coxeter element of $G(d,d,n)$ defined in (4.7). Again Proposition 4.2.4 implies that the claim is true for all Coxeter elements of $G(d,d,n)$.

Finally for the exceptional well-generated complex reflection group, the claim can be checked by computer, for instance using the computer program LINS [81]. $\square$

We conjecture the following property of γ-compatible reflection orders.

CONJECTURE 4.4.28

Let W be a well-generated complex reflection group, and let $\gamma \in W$ be a Coxeter element. If $\prec_\gamma$ is a γ-compatible reflection order of T_γ, then λ_γ is an EL-labeling of $\mathcal{NC}_W(\gamma)$.

Admittedly, the proof of Proposition 4.4.27 is not uniform. However, there might be hope to find a uniform γ-compatible reflection order in our sense. We complete this chapter with another conjecture. Let us first recall some notation.

DEFINITION 4.4.29

Let $\mathcal{P} = (P, \leq)$ be a bounded graded poset with greatest element $\hat{1}$, and let $a_1, a_2, \ldots, a_s$ denote the atoms of $\mathcal{P}$. Then, $\mathcal{P}$ *admits a recursive atom order* if and only if $\mathcal{P}$ has either length 1, or there exists a total order $a_1 \prec a_2 \prec \cdots \prec a_s$ that satisfies the following:

(i) for all $j \in \{1,2,\ldots,s\}$, the interval $[a_j, \hat{1}]$ admits a recursive atom order in which the atoms of $[a_j, \hat{1}]$ that come first in this ordering are those that cover some a_i for $i < j$, and

(ii) for all $i, j \in \{1, 2, \ldots, s\}$ with $i < j$ and some $y \in P$ with $a_i, a_j \leq y$, there exists some $k \in \{1, 2, \ldots, j\}$ and some $z \in P$ with $a_k, a_j \lessdot z \leq y$.

The total order $\prec$ is then called a *recursive atom order*.

CONJECTURE 4.4.30

Let W be a well-generated complex reflection group, and let $\gamma \in W$ be a Coxeter element. Every γ-compatible reflection order of T_γ is a recursive atom order of $\mathcal{NC}_W(\gamma)$.

The importance of recursive atom orders was explained in [24], where it was shown that a graded bounded poset admits a recursive atom order if and only if it is CL-shellable, see [24, Theorem 3.2]. The exact definition of CL-shellability is too technical for the purpose of this section. It was shown that every EL-shellable poset is CL-shellable, and that all the results on EL-shellable posets in Section 1.1.4 also hold for CL-shellable posets. However, it is not known whether both concepts are equivalent. Thus if we could prove Conjecture 4.4.30, the we would obtain a uniform proof of a (possibly) slightly weaker version of Theorem 4.4.1. We conclude this section by proving the first half of Conjecture 4.4.30.

LEMMA 4.4.31

Let $w \in NC_W(\gamma)$ with $\ell_T(w) = 2$, and let $\prec$ denote the restriction of a γ-compatible reflection order of $\mathcal{NC}_W(\gamma)$ to the interval $[\varepsilon, w]$. If $w = rt$ is the unique rising reduced T-decomposition of w, then r is minimal and t is maximal with respect to $\prec$.

PROOF. The fact that $w = rt$ is the unique rising reduced T-decomposition of w follows by definition. Let $r_{\min}$ denote the minimal reflection below w with respect to $\prec$. By definition, there exists a reduced T-decomposition $w = r_{\min}t_1$ for some $t_1 \in T_\gamma \cap [\varepsilon, w]$. Since $r_{\min}$ is minimal it follows that $r_{\min} \prec t_1$ and hence $r = r_{\min}$. Now let $r_{\max}$ denote the maximal reflection below w with respect to $\prec$. Again, by definition, there exists a reduced T-decomposition $w = r_{\max}t_2$ for some $t_2 \in T_\gamma \cap [\varepsilon, w]$. In view of Lemma 4.3.7, there exists another reduced T-decomposition $w = t_3 r_{\max}$, where $t_3 = (r_{\max}^{-1} t_2 r_{\max})$. Since $r_{\max}$ is maximal it follows that $t_3 \prec r_{\max}$ and hence $t = r_{\max}$. □

LEMMA 4.4.32

Let $\prec$ be a γ-compatible reflection order of T_γ. For $w \leq_T \gamma$ the order $\prec$ restricts to a w-compatible reflection order of T_w.

PROOF. Let $w \leq_T \gamma$. We can assume that $\ell_T(w) \geq 2$, because the claim is trivial otherwise. Let $w' \leq_T w$ with $\ell_T(w') = 2$, and let $w' = rt$ be the unique rising reduced T-decomposition of w' with respect to $\prec$, which exists since $w' \leq_T \gamma$ and $\prec$ is γ-compatible. Now, however, it is immediately clear that $w' = rt$ is also the unique rising T-decomposition of w' with respect to the restriction of $\prec$ to $[\varepsilon, w]$ since $T_w \subseteq T_\gamma$. □

PROPOSITION 4.4.33

If $\prec$ is a γ-compatible reflection order of T_γ, then it satisfies Property (i) in Definition 4.4.29.

Proof. We proceed by induction on $\ell_T(\gamma)$. If $\ell_T(\gamma) \leq 2$, then the claim is trivially true. So let $\ell_T(\gamma) \geq 3$, and suppose that the claim is true for all parabolic Coxeter elements $w <_T \gamma$. Let $\prec$ be a γ-compatible reflection order of T_γ, and label the elements of T_γ accordingly, i.e. $T_\gamma = \{t_1, t_2, \ldots, t_N\}$ with $t_i \prec t_j$ if and only if $i < j$. Fix $t_j \in T_\gamma$. It follows from Lemma 4.3.6 that $[t_j, \gamma] \cong [\varepsilon, t_j^{-1}\gamma]$. Write $w = t_j^{-1}\gamma$. In view of Proposition 4.3.4 the element w is a parabolic Coxeter element with $\ell_T(w) < \ell_T(\gamma)$, and Lemma 4.4.32 implies that $\prec$ restricts to a w-compatible reflection order of T_w. Thus by induction hypothesis we conclude that the isomorphism from Lemma 4.3.6 yields a total order of the atoms of $[t_j, \gamma]$, which satisfies Property (i) in Definition 4.4.29, and we will denote this order by $\sqsubset$. Let $a_1, a_2, \ldots, a_s$ denote the atoms of $[t_j, \gamma]$ indexed increasingly with respect to $\sqsubset$. Let $F(j)$ denote the subset of $\{a_1, a_2, \ldots, a_s\}$ consisting of the elements that cover some t_i for $i < j$. We need to show that the elements in $F(j)$ come first in the total order $\sqsubset$, and we proceed by contradiction. Suppose that there are indices $k, l \in \{1, 2, \ldots, s\}$ such that $a_k \sqsubset a_l$, but $a_k \notin F(j)$ and $a_l \in F(j)$. In particular, there exists some $t_i \in T_\gamma$ with $t_i \lessdot_T a_l$ and $t_i \prec t_j$. Since $t_j \lessdot_T a_k$, we can write $a_k = t_j r$ for some $r \in T_\gamma$, and since $a_k \notin F(j)$, we conclude $t_j \prec r$. Analogously, since $t_j \lessdot_T a_l$, we can write $a_l = t_j r'$ for some $r' \in T_\gamma$. Moreover, since $\prec$ is γ-compatible, there exist two unique reflections $r_1, r_2 \in T_\gamma$ with $a_l = r_1 r_2$ and $r_1 \prec r_2$. It follows from Lemma 4.4.31 and from $t_i \prec t_j$ that $r_1 \neq t_j$ which implies $r' \prec t_j$. We conclude from Lemma 4.3.6 that $a_k \sqsubset a_l$ implies $t_j^{-1}a_k \prec t_j^{-1}a_l$. We obtain

$$t_j \prec r = (t_j^{-1}t_j)r = t_j^{-1}(t_j r) = t_j^{-1}a_k \prec t_j^{-1}a_l = t_j^{-1}(t_j r') = (t_j^{-1}t_j)r' = r' \prec t_j,$$

which is a contradiction. Hence the proof is complete. □

The property described in Proposition 4.4.33 can be seen as an CL-shellability-analogue of Proposition 4.3.9.

CHAPTER 5

Epilogue

5.1. Conclusion

In this thesis we have investigated the structure and the topology of several different classes of posets. In particular, we have considered the m-Tamari lattices, the Cambrian semilattices and the lattices of noncrossing partitions. These posets belong to two closely related frameworks: on the one hand, they are in some way associated with Coxeter groups, and on the other hand their cardinality is (in the finite case) given by the Fuß-Catalan numbers. In particular, the classical Fuß-Catalan numbers are intrinsically connected to the Coxeter group of type A. More precisely, we have $\mathrm{Cat}^{(m)}(n) = \mathrm{Cat}^{(m)}(A_{n-1})$.

From a topological point of view we have shown that these posets have the homotopy type of a wedge of spheres, see Theorems 2.3.1, 3.4.1, and 4.4.1, and thus a particularly nice topological structure. For some special cases, these results were already available in the literature, and we could extend these results to the most general case. We have obtained our results by defining a special edge-labeling for each of these classes of posets. Subsequently, we have shown that these labelings are EL-labelings, which then implied the claims. We remark that these edge-labelings are uniform in the sense that they do not rely on the classification of Coxeter groups. However, for the lattices of noncrossing partitions, the proof that the corresponding labeling is an EL-labeling is not obtained in a uniform way, but required a case-by-case analysis instead. This case-by-case analysis, on the other hand, strongly suggests that there is a uniform approach to Theorem 4.4.1, see Conjecture 4.4.28. Subsequently, we have used these edge-labelings for the computation of the Möbius function for each of these classes of posets.

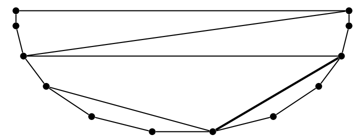
We have additionally investigated the Cambrian semilattices from a structural point of view, and our results can be transferred immediately to the m-Tamari lattices since the latter can be seen as intervals in some special Cambrian semilattices. In particular, we have shown that each closed interval in a Cambrian semilattice is a bounded-homomorphic image of a free lattice, see Theorem 3.5.1. It is an immediate consequence of this result that each such interval can be constructed from the one-element lattice by a series of consecutive interval doublings. We have obtained this result by means of another edge-labeling that has been defined uniformly for all Cambrian semilattices. This edge-labeling indicates precisely what the sequence of interval doublings looks like.

5.2. Future Work

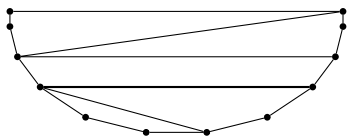
Let us now restrict our attention to finite Coxeter groups. It follows from Remark 3.2.2 that the family of the m-Tamari lattices and the family of the Cambrian lattices both have a common member: the Tamari lattices. It is well known that the cardinality of the Tamari lattice $\mathcal{T}_n$ is given by the Catalan number $\mathrm{Cat}(n)$. The m-Tamari lattices and the Cambrian lattices can be seen as two orthogonal generalizations of the Tamari lattice in the following sense: the cardinality of the m-Tamari lattice $\mathcal{T}_n^{(m)}$ is given by the Fuß-Catalan number $\mathrm{Cat}^{(m)}(n)$, and the cardinality of the Cambrian lattice C_γ associated with a finite Coxeter group W is given (for all Coxeter elements $\gamma \in W$) by the Coxeter-Catalan number $\mathrm{Cat}(W)$. It is now an immediate question whether we can unify both generalizations towards a family of posets parametrized by a Coxeter group W and a positive integer m such that their cardinality is given by the corresponding Coxeter-Fuß-Catalan number $\mathrm{Cat}^{(m)}(W)$. Moreover, such a generalization should yield the Cambrian lattices in the case $m = 1$, and it should yield the m-Tamari lattices in the case $W = A_n$. We have presented an approach towards such a generalization in Section 2.4.4 using the m-cover posets, and this approach worked beautifully for the dihedral groups. However, we were unable to generalize this to other Coxeter groups. In this section, we discuss another approach towards this generalization.

5.2.1. Generalized Flip Posets of Triangulations. Recall from Section 3.1 that Reading originally defined the Cambrian lattices as flip posets on triangulations of convex polygons. Recall further that there is a classical generalization of triangulations of convex polygons that involves the Fuß-Catalan numbers. It was observed by Fuss in [**126**] that the number of dissections of a convex $(mn+2)$-gon into $(m+2)$-gons is given by $\mathrm{Cat}^{(m)}(n)$. We mimic Reading's construction of convex $(mn+2)$-gons using a map $f : \{1,2,\ldots,mn\} \to \{-1,1\}$. We draw the vertices 0 and $mn+1$ on a horizontal line, the *horizon*, and for $i \in \{1,2,\ldots,mn\}$ we place the vertex i strictly between the vertices $i-1$ and $i+1$. We place it below the horizon if and only if $f(i) = -1$ and above otherwise. We can make sure that this construction yields a convex $(mn+2)$-gon in which no $m+2$ vertices are collinear. We denote this polygon by $Q_{mn}^{(f)}$, and we denote the set of dissections of $Q_{mn}^{(f)}$ into $(m+2)$-gons by $\Delta^{(m)}\bigl(Q_{mn}^{(f)}\bigr)$. Again a dissection $D \in \Delta^{(m)}\bigl(Q_{mn}^{(f)}\bigr)$ is completely determined by its *diagonals*, namely lines connecting two vertices i and j with $i \not\equiv j \pm 1 \pmod{mn+2}$. In contrast to the case of triangulations, removing a diagonal d from D yields a $(2m+2)$-gon, and we have in general more than one possibility of inserting another diagonal such that we obtain a different dissection $D' \in \Delta^{(m)}\bigl(Q_{mn}^{(f)}\bigr)$. The two most natural choices are either "sliding" the diagonal d one step to the left (i.e. clockwise) or one step to the right (i.e. counter-clockwise). See Figure 47 for an example. We observe that in the case $m = 1$ both of these operations coincide with Reading's flip operation described in Section 3.1.

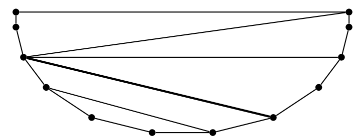
Now we can again construct partial orders $\leq_{\mathrm{left}}$ and $\leq_{\mathrm{right}}$ on $\Delta^{(m)}\bigl(Q_{mn}^{(f)}\bigr)$ whose cover relations are given by left or right-slides, respectively. For the right-slide posets we recover the property that going up means increasing the slope, however it is not yet clear how to formalize this property for the left-slide posets. Figure 48 shows the posets $\Bigl(\Delta^{(2)}\bigl(Q_6^{(f)}\bigr), \leq_{\mathrm{left}}\Bigr)$ and $\Bigl(\Delta^{(2)}\bigl(Q_6^{(f)}\bigr), \leq_{\mathrm{right}}\Bigr)$, where $f(i) = -1$ for $i \in \{1,2,\ldots,6\}$. We notice that none of these posets is isomorphic to the 2-Tamari lattice $\mathcal{T}_3^{(2)}$ as displayed in Figure 49. In fact, only the left-slide poset in Figure 48(a) is a lattice, while the right-slide poset is not. However, these



Right-sliding the highlighted diagonal.



A triangulation $D \in \Delta^{(2)}(Q_6)$.



Left-sliding the highlighted diagonal.

FIGURE 47. Illustration of diagonal slides.

posets share many enumerative properties with the 2-Tamari lattice $\mathcal{T}_3^{(2)}$. We conclude the following open problem.

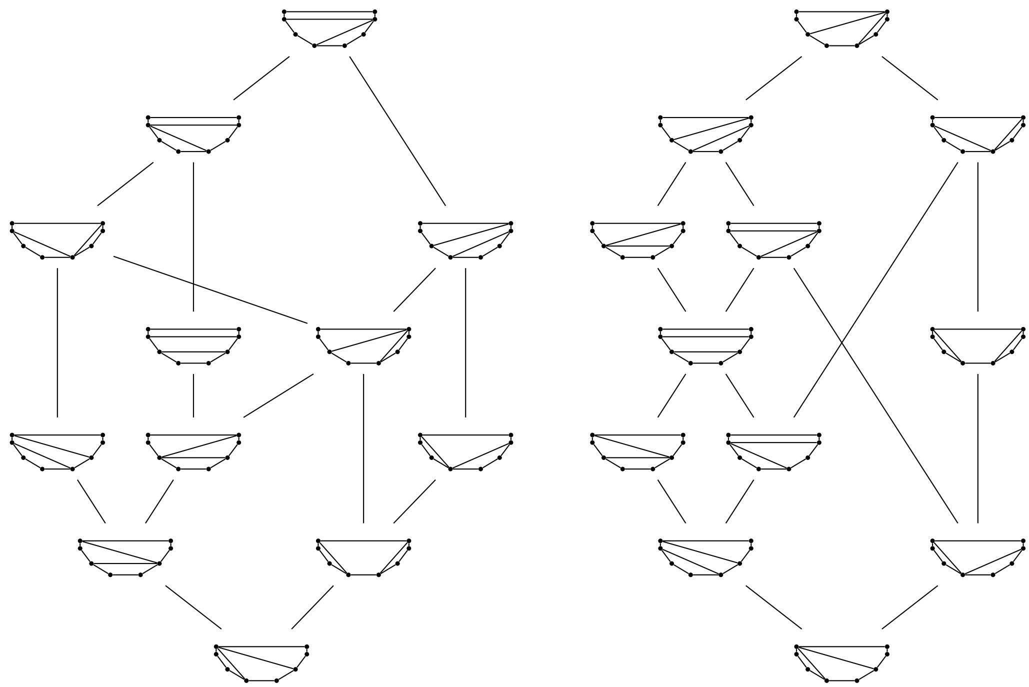
OPEN PROBLEM 5.2.1

Investigate the posets $\left(\Delta^{(m)}(Q_{mn}^{(f)}), \leq_{\text{left}}\right)$ and $\left(\Delta^{(m)}(Q_{mn}^{(f)}), \leq_{\text{right}}\right)$ for the various choices of maps $f : \{1,2,\ldots,mn\} \to \{-1,1\}$. In particular, investigate their connection to the m-Tamari lattices $\mathcal{T}_n^{(m)}$.

In Figure 50, the left- and the right-slide posets of $\Delta^{(2)}(Q_6^{(f')})$ for $f'(1) = f'(2) = f'(3) = 1$ and $f'(4) = f'(5) = f'(6) = -1$ are shown. We notice that these posets are isomorphic, and they share some enumerative properties of $\mathcal{T}_3^{(2)}$, but not all of them. The table in Figure 51 lists the similarities and the differences of the posets discussed in this section.

5.2.2. Connection to Generalized Cluster Complexes. The sliding posets from the previous section might also have a close connection to the generalized cluster complexes, defined by FOMIN and READING in [53]. Let us first consider the case $m = 1$. In his thesis, STASHEFF realized the associahedron as a complex whose vertices are triangulations of a convex polygon and whose facets are diagonals of this same polygon, [114]. The dual of this complex was later generalized to finite Weyl groups by FOMIN and ZELEVINSKY as the so-called cluster complex, see [54].

Hence the cluster complex associated with the Coxeter group of type A is the dual of the associahedron, and it can thus be seen as a simplicial complex whose vertices are diagonals of a convex polygon. A Fuß-Catalan generalization of this complex was later given by REINER in type A and by ATHANASIADIS in type B, and it was thoroughly studied by TZANAKI, see [124]. Not long after that FOMIN and READING defined in [53] m-cluster complexes for all Weyl groups, which include the previously mentioned complexes. One remarkable property that all of these complexes have in common is that the cardinality of their vertices is given by the Fuß-Catalan number associated with the Weyl group for which this complex was defined.



(a) The poset $\left(\Delta^{(2)}(Q_6^{(f)}), \leq_{\text{left}}\right)$.

(b) The poset $\left(\Delta^{(2)}(Q_6^{(f)}), \leq_{\text{right}}\right)$.

FIGURE 48. The left- and right-slide posets of $\Delta^{(2)}(Q_6^{(f)})$, where $f(i) = -1$ for $i \in \{1, 2, \ldots, 6\}$.

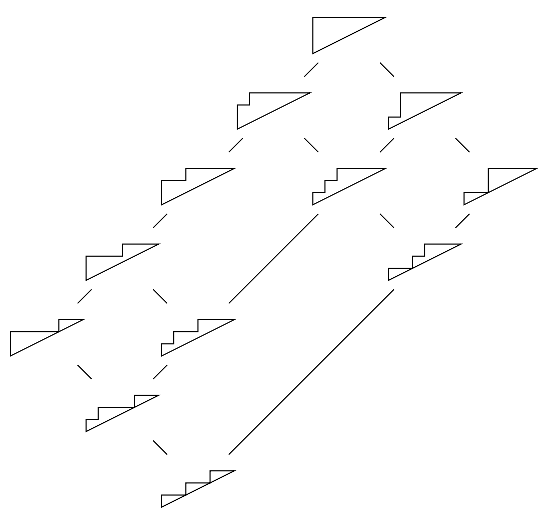
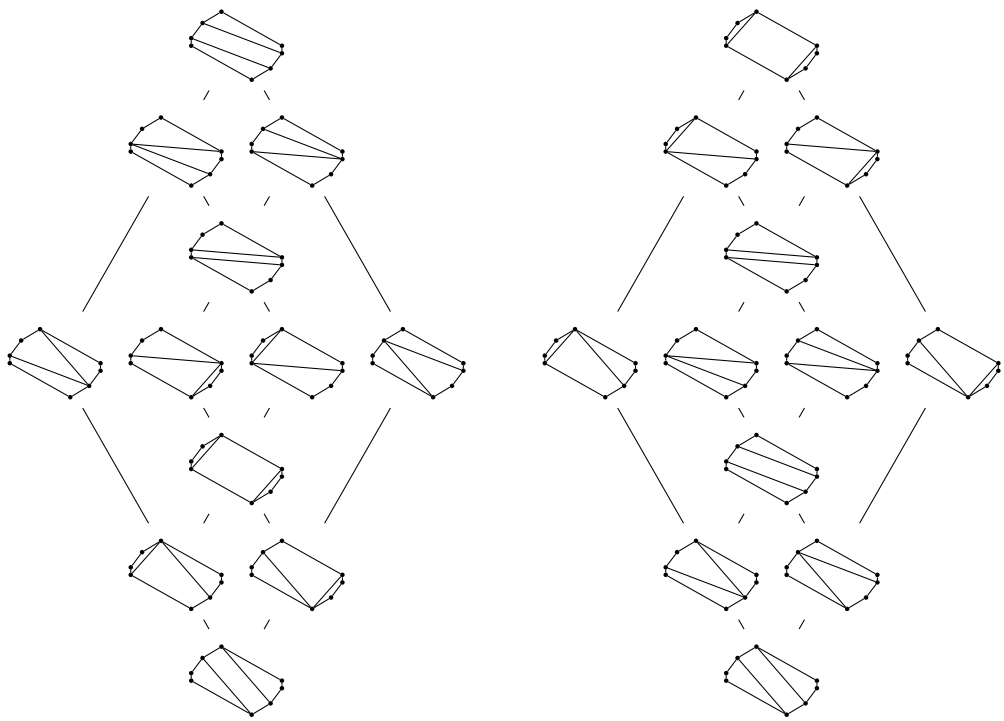


FIGURE 49. The 2-Tamari lattice $\mathcal{T}_3^{(2)}$ again.



(a) The poset $\left(\Delta^{(2)}\left(Q_6^{(f')}\right), \leq_{\text{left}}\right)$.

(b) The poset $\left(\Delta^{(2)}\left(Q_6^{(f')}\right), \leq_{\text{right}}\right)$.

FIGURE 50. The left- and right-slide posets of $\Delta^{(2)}\left(Q_6^{(f')}\right)$, where $f'(1) = f'(2) = f'(3) = 1$ and $f'(4) = f'(5) = f'(6) = -1$.

	$\mathcal{T}_3^{(2)}$	$\left(\Delta^{(2)}\left(Q_6^{(f)}\right), \leq_{\text{left}}\right)$	$\left(\Delta^{(2)}\left(Q_6^{(f)}\right), \leq_{\text{right}}\right)$	$\left(\Delta^{(2)}\left(Q_6^{(f')}\right), \leq_{\text{left}}\right)$
lattice	yes	yes	no	yes
vertices	12	12	12	12
edges	16	16	16	16
intervals	58	58	59	62
Möbius values	$\{-1, 0, 1\}$	$\{-1, 0, 1\}$	$\{-1, 0, 1\}$	$\{-1, 0, 1\}$
Möbius vector	$(16, 111, 17)$	$(16, 111, 17)$	$(17, 109, 18)$	$(16, 111, 17)$
cover vector	$(1, 6, 5)$	$(1, 6, 5)$	$(1, 6, 5)$	$(1, 6, 5)$

FIGURE 51. Comparing different sliding posets with the m-Tamari lattice for $m = 2$ and $n = 3$. For $j \in \{1, 2, 3\}$, the j-th entry of the Möbius vector corresponds to the number of pairs (p, q) in the corresponding poset with $\mu(p, q) = j - 2$, and the j-th entry of the cover vector corresponds to the number of elements in the corresponding poset with $j - 1$ upper covers.

The m-cluster complex of type A can be realized as a simplicial complex whose vertices are diagonals of some polygon, and whose facets are $(m + 2)$-angulations of this same polygon. Hence its dual would be a complex having $(m + 2)$-angulations of some polygon as vertices.

An extensive overview on the combinatorial properties of these m-cluster complexes can be found in [1, Section 5.2].

The previously described construction suggests an intriguing connection between the duals of the m-cluster complexes and the sliding posets introduced in the previous section. Since we have seen that the lattice $\mathcal{T}_3^{(2)}$ does not arise as a sliding poset of quadrangulations of a convex hexagon, we conclude that the m-Tamari lattice might not be included in a suitable Fuß-Catalan-generalization of the family of Cambrian lattices associated with finite Coxeter groups. We end this thesis with the following open problem.

OPEN PROBLEM 5.2.2

Define "m-Tamari like" lattices for all finite Coxeter groups such that in type A one obtains the m-Tamari lattices, and in type I one obtains our lattices $\mathcal{C}_k^{\langle m\rangle}$ from Section 2.4.4.

APPENDIX A

Tables for the Cardinalities of m-Cover Posets of Some Cambrian Lattices and Their Lattice Completions

A.1. Type A

EXAMPLE A.1.1

Let $W = A_3$ with Coxeter diagram $s_1 — s_2 — s_3$. The table in Figure 52 lists the cardinalities of $\mathbf{DM}\Big(\big(\mathcal{C}_\gamma^{\langle m\rangle}, \leq_\gamma\big)\Big)$ for $m \in \{1,2,3,4\}$. The sequence of m-Catalan numbers for A_3 starts with

$$14, 55, 140, 285, \ldots.$$

EXAMPLE A.1.2

Let $W = A_4$ with Coxeter diagram $s_1 — s_2 — s_3 — s_4$. The table in Figure 53 lists the cardinalities of $\mathbf{DM}\Big(\big(\mathcal{C}_\gamma^{\langle m\rangle}, \leq_\gamma\big)\Big)$ for $m \in \{1,2,3,4\}$. The sequence of m-Catalan numbers for A_4 starts with

$$42, 273, 969, 2530, \ldots.$$

A.2. Type B

	$m=1$	$m=2$	$m=3$	$m=4$
$\gamma = s_1s_2s_3$	14	55	140	285
$\gamma = s_1s_3s_2$	14	56	146	305
$\gamma = s_2s_1s_3$	14	59	162	355
$\gamma = s_3s_2s_1$	14	55	140	285
$\lvert\mathcal{C}_\gamma^{\langle m\rangle}\rvert$	14	45	94	161

FIGURE 52. The cardinalities of $\mathbf{DM}\Big(\big(\mathcal{C}_\gamma^{\langle m\rangle}, \leq_\gamma\big)\Big)$ for $W = A_3$ and $m \in \{1,2,3,4\}$.

	$m=1$	$m=2$	$m=3$	$m=4$
$\gamma = s_1s_2s_3s_4$	42	273	969	2530
$\gamma = s_1s_3s_2s_4$	42	308	1252	3741
$\gamma = s_1s_4s_2s_3$	42	282	1045	2860
$\gamma = s_1s_4s_3s_2$	42	282	1045	2860
$\gamma = s_2s_3s_1s_4$	42	305	1211	3510
$\gamma = s_3s_2s_1s_4$	42	305	1211	3510
$\gamma = s_4s_2s_3s_1$	42	308	1252	3741
$\gamma = s_4s_3s_2s_1$	42	273	969	2530
$\lvert C_\gamma^{\langle m\rangle}\rvert$	42	163	364	645

FIGURE 53. The cardinalities of $\mathbf{DM}\Big(\big(C_\gamma^{\langle m\rangle}, \leq_\gamma\big)\Big)$ for $W = A_4$ and $m \in \{1,2,3,4\}$.

	$m=1$	$m=2$	$m=3$	$m=4$
$\gamma = s_1s_2s_3$	20	88	242	525
$\gamma = s_1s_3s_2$	20	89	249	550
$\gamma = s_2s_1s_3$	20	97	292	685
$\gamma = s_3s_2s_1$	20	85	226	475
$\lvert C_\gamma^{\langle m\rangle}\rvert$	20	66	139	239

FIGURE 54. The cardinalities of $\mathbf{DM}\Big(\big(C_\gamma^{\langle m\rangle}, \leq_\gamma\big)\Big)$ for $W = B_3$ and $m \in \{1,2,3,4\}$.

EXAMPLE A.2.1

Let $W = B_3$ with Coxeter diagram $s_1 - s_2 \overset{4}{-} s_3$. The table in Figure 54 lists the cardinalities of $\mathbf{DM}\Big(\big(C_\gamma^{\langle m\rangle}, \leq_\gamma\big)\Big)$ for $m \in \{1,2,3,4\}$. The sequence of m-Catalan numbers for B_3 starts with

$$20, 84, 220, 455, \ldots.$$

EXAMPLE A.2.2

Let $W = B_4$ with Coxeter diagram $s_1 - s_2 - s_3 \overset{4}{-} s_4$. The table in Figure 55 lists the cardinalities of $\mathbf{DM}\Big(\big(C_\gamma^{\langle m\rangle}, \leq_\gamma\big)\Big)$ for $m \in \{1,2,3,4\}$. The sequence of m-Catalan numbers for B_4 starts with

$$70, 495, 1820, 4845, \ldots.$$

A.3. Type D

EXAMPLE A.3.1

Let $W = D_4$ with Coxeter diagram $\begin{array}{ccccc} & & s_4 & & \\ & & \vert & & \\ s_1 & - & s_3 & - & s_2 \end{array}$. The table in Figure 56 lists the cardinalities of $\mathbf{DM}\Big(\big(C_\gamma^{\langle m\rangle}, \leq_\gamma\big)\Big)$ for $m \in \{1,2,3,4\}$. The sequence of m-Catalan numbers for D_4

	$m=1$	$m=2$	$m=3$	$m=4$
$\gamma = s_1s_2s_3s_4$	70	547	2216	6455
$\gamma = s_1s_3s_2s_4$	70	619	2870	9504
$\gamma = s_1s_4s_2s_3$	70	558	2339	7085
$\gamma = s_1s_4s_3s_2$	70	555	2330	7099
$\gamma = s_2s_3s_1s_4$	70	643	3023	10035
$\gamma = s_3s_2s_1s_4$	70	598	2642	8320
$\gamma = s_4s_2s_3s_1$	70	630	2959	9886
$\gamma = s_4s_3s_2s_1$	70	510	1948	5405
$\lvert C_\gamma^{\langle m\rangle}\rvert$	70	275	616	1093

FIGURE 55. The cardinalities of $\mathbf{DM}\Big(\big(C_\gamma^{\langle m\rangle}, \leq_\gamma\big)\Big)$ for $W = B_4$ and $m \in \{1,2,3,4\}$.

	$m=1$	$m=2$	$m=3$	$m=4$
$\gamma = s_1s_2s_3s_4$	50	358	1382	3895
$\gamma = s_1s_3s_2s_4$	50	381	1574	4741
$\gamma = s_1s_4s_2s_3$	50	394	1761	5936
$\gamma = s_1s_4s_3s_2$	50	358	1382	3895
$\gamma = s_2s_3s_1s_4$	50	381	1574	4741
$\gamma = s_3s_2s_1s_4$	50	445	2135	7353
$\gamma = s_4s_2s_3s_1$	50	358	1382	3895
$\gamma = s_4s_3s_2s_1$	50	381	1574	4741
$\lvert C_\gamma^{\langle m\rangle}\rvert$	50	195	436	773

FIGURE 56. The cardinalities of $\mathbf{DM}\Big(\big(C_\gamma^{\langle m\rangle}, \leq_\gamma\big)\Big)$ for $W = D_4$ and $m \in \{1,2,3,4\}$.

starts with
$$50, 336, 1210, 3185, \ldots.$$

A.4. Type F

EXAMPLE A.4.1

Let $W = F_4$ with Coxeter diagram $s_1 \frac{\quad}{} s_2 \frac{4}{\quad} s_3 \frac{\quad}{} s_4$. The table in Figure 57 lists the cardinalities of $\mathbf{DM}\Big(\big(C_\gamma^{\langle m\rangle}, \leq_\gamma\big)\Big)$ for $m \in \{1,2,3,4\}$. The sequence of m-Catalan numbers for F_4 starts with
$$105, 780, 2926, 7875, \ldots.$$

A.5. Type H

EXAMPLE A.5.1

Let $W = H_3$ with Coxeter diagram $s_1 \frac{\quad}{} s_2 \frac{5}{\quad} s_3$. The table in Figure 58 lists the cardinalities of $\mathbf{DM}\Big(\big(C_\gamma^{\langle m\rangle}, \leq_\gamma\big)\Big)$. The sequence of m-Catalan numbers for H_3 starts with
$$32, 143, 384, 805, \ldots.$$

	$m=1$	$m=2$	$m=3$	$m=4$
$\gamma = s_1s_2s_3s_4$	105	960	4497	15062
$\gamma = s_1s_3s_2s_4$	105	1218	7280	30545
$\gamma = s_1s_4s_2s_3$	105	1065	5754	22561
$\gamma = s_1s_4s_3s_2$	105	1065	5754	22561
$\gamma = s_2s_3s_1s_4$	105	1192	6666	25687
$\gamma = s_3s_2s_1s_4$	105	1192	6666	25687
$\gamma = s_4s_2s_3s_1$	105	1218	7280	30545
$\gamma = s_4s_3s_2s_1$	105	960	4497	15062
$\lvert C_\gamma^{\langle m\rangle}\rvert$	105	415	931	1653

FIGURE 57. The cardinalities of $\mathbf{DM}\Big(\big(C_\gamma^{\langle m\rangle}, \leq_\gamma\big)\Big)$ for $W = F_4$ and $m \in \{1,2,3,4\}$.

	$m=1$	$m=2$	$m=3$	$m=4$
$\gamma = s_1s_2s_3$	32	165	506	1195
$\gamma = s_1s_3s_2$	32	165	515	1248
$\gamma = s_2s_1s_3$	32	184	622	1598
$\gamma = s_3s_2s_1$	32	152	436	975
$\lvert C_\gamma^{\langle m\rangle}\rvert$	32	108	229	395

FIGURE 58. The cardinalities of $\mathbf{DM}\Big(\big(C_\gamma^{\langle m\rangle}, \leq_\gamma\big)\Big)$ for $W = H_3$ and $m \in \{1,2,3,4\}$.

	$m=1$	$m=2$	$m=3$	$m=4$
$\gamma = s_1s_2s_3s_4$	280	4034	26659	117284
$\gamma = s_1s_3s_2s_4$	280	4407	32502	159432
$\gamma = s_1s_4s_2s_3$	280	4074	28649	136802
$\gamma = s_1s_4s_3s_2$	280	4148	30861	158395
$\gamma = s_2s_3s_1s_4$	280	4792	37444	191441
$\gamma = s_3s_2s_1s_4$	280	4264	30813	149662
$\gamma = s_4s_2s_3s_1$	280	4630	37529	205236
$\gamma = s_4s_3s_2s_1$	280	3676	24213	111275
$\lvert C_\gamma^{\langle m\rangle}\rvert$	280	1115	2506	4453

FIGURE 59. The cardinalities of $\mathbf{DM}\Big(\big(C_\gamma^{\langle m\rangle}, \leq_\gamma\big)\Big)$ for $W = H_4$ and $m \in \{1,2,3,4\}$.

EXAMPLE A.5.2

Let $W = H_4$ with Coxeter diagram $s_1 - s_2 - s_3 \overset{5}{-} s_4$. The table in Figure 59 lists the cardinalities of $\mathbf{DM}\Big(\big(C_\gamma^{\langle m\rangle}, \leq_\gamma\big)\Big)$ for $m \in \{1,2,3,4\}$. The sequence of m-Catalan numbers for H_4 starts with

$$280, 2232, 8602, 23485, \ldots.$$

APPENDIX B

Deconstructing $\mathcal{T}_4$ by Successive Interval Contracting

Figure 60 shows how to deconstruct $\mathcal{T}_4$ by successive contraction of intervals. These contractions are made with respect to the edge-labeling defined in (3.10). We successively contract those intervals of $\mathcal{T}_4$ in which the longest edge-labels occur. This means that we remove the edges with these longest labels from the Hasse diagram of $\mathcal{T}_4$ and we identify "opposite" edges, which by construction have the same labels.

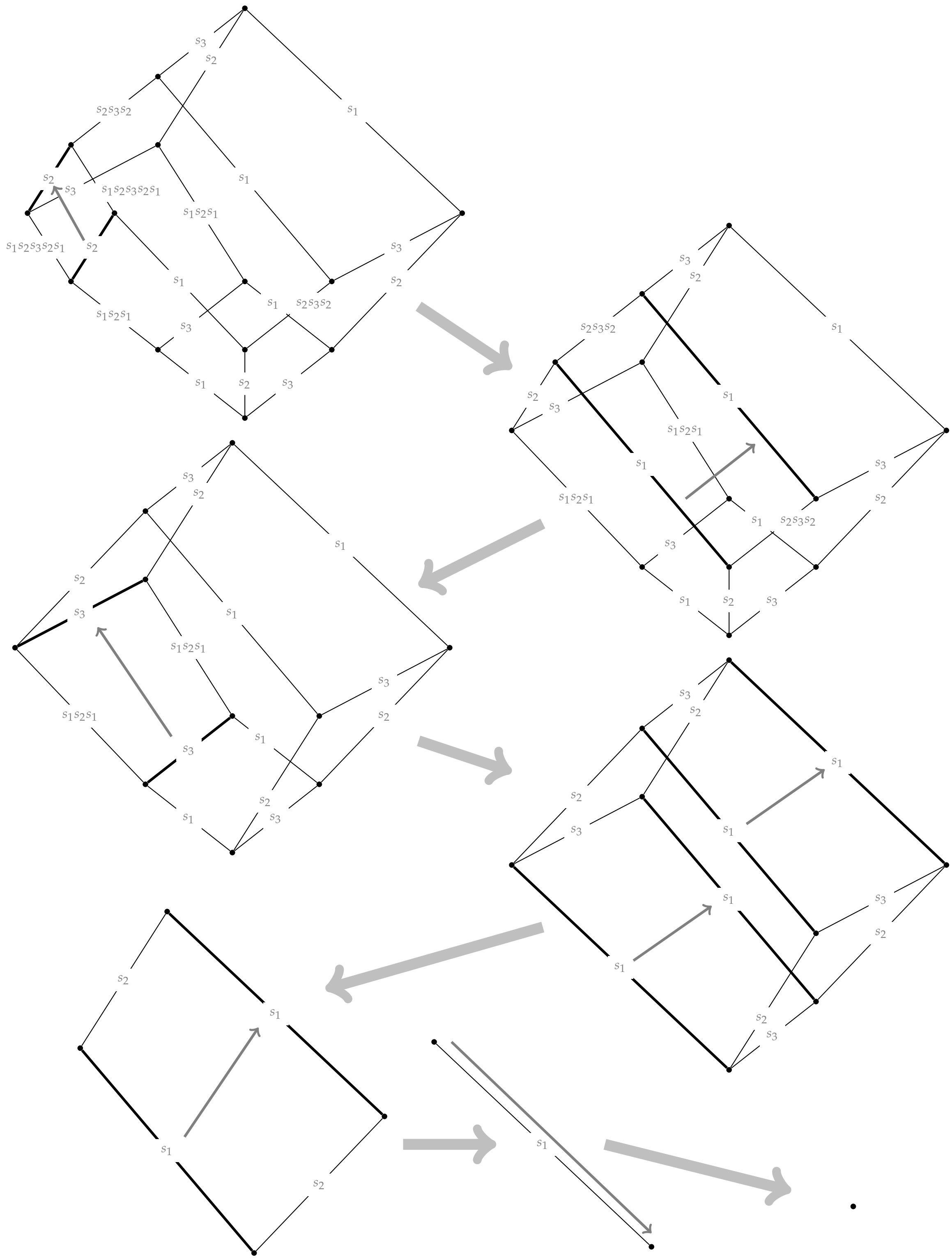


FIGURE 60. Contracting Intervals of $\mathcal{T}_4$.

APPENDIX C

The Details of the Proofs in Section 4.4.1

The proofs of the intermediate steps leading to the proof of Theorem 4.4.11 are rather technical, and we have thus omitted the details in the text. For the sake of completeness, we provide the details in this appendix, after recalling the respective statements.

C.1. The Proof of Lemma 4.4.15

Lemma C.1.1

Let $w \leq_T \gamma$ with $\ell_T(w) = 2$. There exists a unique rising reduced T-decomposition of w with respect to the restriction of $\prec_\gamma$ to the reflections in $T_\gamma \cap [\varepsilon, w]$.

Proof. Let $w = t_1 t_2$ for $t_1, t_2 \in T_\gamma$. If t_1 and t_2 commute, then $w = t_1 t_2 = t_2 t_1$ are the only possible reduced T-decompositions of w. Since $\prec_\gamma$ is a total order there is nothing to show. Suppose that t_1 and t_2 do not commute. With the help of Proposition 4.4.8, we can explicitly determine the possible forms of w. Analogously to the proof of Proposition 4.4.8, we investigate the fixed space of $w^{-1}\gamma$ to determine which of these possibilities can actually occur in $\mathcal{NC}_{G(d,d,n)}(\gamma)$. Recall from (4.12) that for an arbitrary vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)^\mathsf{T} \in \mathbb{C}^n$, we have

$$\mathbf{v}' = \gamma\mathbf{v} = \left(\zeta_d v_{n-1}, v_1, v_2, \ldots, v_{n-2}, \zeta_d^{d-1} v_n\right)^\mathsf{T}.$$

(i) Let $t_1 = \left(\left(a^{(0)}\ b^{(0)}\right)\right), t_2 = \left(\left(b^{(0)}\ c^{(0)}\right)\right)$, where $1 \leq a < b < c < n$. We have $w = \left(\left(a^{(0)}\ b^{(0)}\ c^{(0)}\right)\right)$, and thus

$$w^{-1}\mathbf{v}' = \left(\zeta_d v_{n-1}, v_1, \ldots, v_{b-1}, \ldots, v_{c-1}, \ldots, v_{a-1}, \ldots, v_{n-2}, \zeta_d^{d-1} v_n\right)^\mathsf{T},$$

and it follows that $w \leq_T \gamma$. Hence $w^{-1} = \left(\left(a^{(0)}\ c^{(0)}\ b^{(0)}\right)\right) \not\leq_T \gamma$. The reduced T-decompositions of w are

$$w = \left(\left(a^{(0)}\ b^{(0)}\right)\right)\left(\left(b^{(0)}\ c^{(0)}\right)\right) = \left(\left(b^{(0)}\ c^{(0)}\right)\right)\left(\left(a^{(0)}\ c^{(0)}\right)\right) = \left(\left(a^{(0)}\ c^{(0)}\right)\right)\left(\left(a^{(0)}\ b^{(0)}\right)\right),$$

and according to (4.13) only $w = \left(\left(a^{(0)}\ b^{(0)}\right)\right)\left(\left(b^{(0)}\ c^{(0)}\right)\right)$ is increasing.

(ii) Let $t_1 = \left(\left(a^{(0)}\ b^{(0)}\right)\right), t_2 = \left(\left(a^{(0)}\ c^{(0)}\right)\right)$, where $1 \leq a < b < c < n$. We have $w = \left(\left(a^{(0)}\ c^{(0)}\ b^{(0)}\right)\right)$, and this was already considered in (i).

(iii) Let $t_1 = \big(\!\big(a^{(0)}\ b^{(0)}\big)\!\big), t_2 = \big(\!\big(a^{(0)}\ b^{(d-1)}\big)\!\big)$, where $1 \leq a < b < n$. We have $w = \big[a^{(0)}\big]\big[b^{(0)}\big]^{-1}$, and

$$w^{-1}\mathbf{v}' = \Big(\zeta_d v_{n-1}, v_1, \ldots, \zeta_d v_{a-1}, \ldots, \zeta_d^{d-1} v_{b-1}, \ldots, v_{n-2}, \zeta_d^{d-1} v_n\Big)^{\mathsf{T}},$$

and it follows that $w \not\leq_T \gamma$. On the other hand we have

$$w\mathbf{v}' = \Big(\zeta_d v_{n-1}, v_1, \ldots, \zeta_d^{d-1} v_{a-1}, \ldots, \zeta_d v_{b-1}, \ldots, v_{n-2}, \zeta_d^{d-1} v_n\Big)^{\mathsf{T}},$$

and it follows again that $w^{-1} \not\leq_T \gamma$.

(iv) Let $t_1 = \big(\!\big(a^{(0)}\ b^{(0)}\big)\!\big), t_2 = \big(\!\big(b^{(0)}\ c^{(d-1)}\big)\!\big)$, where $1 \leq a < b < c < n$. We have $w = \big(\!\big(a^{(0)}\ b^{(0)}\ c^{(d-1)}\big)\!\big)$, and

$$w^{-1}\mathbf{v}' = \Big(\zeta_d v_{n-1}, v_1, \ldots, v_{b-1}, \ldots, \zeta_d v_{c-1}, \ldots, \zeta_d^{d-1} v_{a-1}, \ldots, v_{n-2}, \zeta_d^{d-1} v_n\Big)^{\mathsf{T}},$$

and it follows that $w \leq_T \gamma$. Hence $w^{-1} = \big(\!\big(a^{(0)}\ c^{(d-1)}\ b^{(0)}\big)\!\big) \not\leq_T \gamma$. The reduced T-decompositions of w are

$$w = \big(\!\big(a^{(0)}\ b^{(0)}\big)\!\big)\big(\!\big(b^{(0)}\ c^{(d-1)}\big)\!\big) = \big(\!\big(b^{(0)}\ c^{(d-1)}\big)\!\big)\big(\!\big(a^{(0)}\ c^{(d-1)}\big)\!\big) = \big(\!\big(a^{(0)}\ c^{(d-1)}\big)\!\big)\big(\!\big(a^{(0)}\ b^{(0)}\big)\!\big),$$

and according to (4.13) only $w = \big(\!\big(a^{(0)}\ b^{(0)}\big)\!\big)\big(\!\big(b^{(0)}\ c^{(d-1)}\big)\!\big)$ is increasing.

(v) Let $t_1 = \big(\!\big(a^{(0)}\ b^{(0)}\big)\!\big), t_2 = \big(\!\big(a^{(0)}\ c^{(d-1)}\big)\!\big)$, where $1 \leq a < b < c < n$. We have $w = \big(\!\big(a^{(0)}\ c^{(d-1)}\ b^{(0)}\big)\!\big)$, and this was already considered in (iv).

(vi) Let $t_1 = \big(\!\big(a^{(0)}\ b^{(0)}\big)\!\big), t_2 = \big(\!\big(b^{(0)}\ n^{(s)}\big)\!\big)$, where $1 \leq a < b < n$ and $0 \leq s < d$. We have $w = \big(\!\big(a^{(0)}\ b^{(0)}\ n^{(s)}\big)\!\big)$, and

$$w^{-1}\mathbf{v}' = \Big(\zeta_d v_{n-1}, v_1, \ldots, v_{b-1}, \ldots, \zeta_d^{d-1-s} v_n, \ldots, v_{n-2}, \zeta_d^{s} v_{a-1}\Big)^{\mathsf{T}},$$

and it follows that $w \leq_T \gamma$. Hence $w^{-1} = \big(\!\big(a^{(0)}\ n^{(s)}\ b^{(0)}\big)\!\big) \not\leq_T \gamma$. The reduced T-decompositions of w are

$$w = \big(\!\big(a^{(0)}\ b^{(0)}\big)\!\big)\big(\!\big(b^{(0)}\ n^{(s)}\big)\!\big) = \big(\!\big(b^{(0)}\ n^{(s)}\big)\!\big)\big(\!\big(a^{(0)}\ n^{(s)}\big)\!\big) = \big(\!\big(a^{(0)}\ n^{(s)}\big)\!\big)\big(\!\big(a^{(0)}\ b^{(0)}\big)\!\big),$$

and according to (4.13) only $w = \big(\!\big(a^{(0)}\ b^{(0)}\big)\!\big)\big(\!\big(b^{(0)}\ n^{(s)}\big)\!\big)$ is increasing.

(vii) Let $t_1 = \big(\!\big(a^{(0)}\ b^{(0)}\big)\!\big), t_2 = \big(\!\big(a^{(0)}\ n^{(s)}\big)\!\big)$, where $1 \leq a < b < n$ and $0 \leq s < d$. We have $w = \big(\!\big(a^{(0)}\ n^{(s)}\ b^{(0)}\big)\!\big)$, and this was already considered in (vi).

(viii) Let $t_1 = \big(\!\big(a^{(0)}\ c^{(0)}\big)\!\big), t_2 = \big(\!\big(b^{(0)}\ c^{(0)}\big)\!\big)$, where $1 \leq a < b < c < n$. We have $w = \big(\!\big(a^{(0)}\ c^{(0)}\ b^{(0)}\big)\!\big)$, and

$$w^{-1}\mathbf{v}' = \Big(\zeta_d v_{n-1}, v_1, \ldots, v_{c-1}, \ldots, v_{a-1}, \ldots, v_{b-1}, \ldots, v_{n-2}, \zeta_d^{d-1} v_n\Big)^{\mathsf{T}},$$

and it follows that $w \not\leq_T \gamma$. On the other hand $w^{-1} = \big(\!\big(a^{(0)}\ b^{(0)}\ c^{(0)}\big)\!\big)$ was considered in (i).

(ix) Let $t_1 = \big(\!\big(a^{(0)}\ c^{(0)}\big)\!\big), t_2 = \big(\!\big(a^{(0)}\ b^{(d-1)}\big)\!\big)$, where $1 \leq a < b < c < n$. We have $w = \big(\!\big(a^{(0)}\ b^{(d-1)}\ c^{(0)}\big)\!\big)$, and

$$w^{-1}\mathbf{v}' = \Big(\zeta_d v_{n-1}, v_1, \ldots, \zeta_d v_{b-1}, \ldots, \zeta_d^{d-1} v_{c-1}, \ldots, v_{a-1}, \ldots, v_{n-2}, \zeta_d^{d-1} v_n\Big)^{\mathsf{T}},$$

and it follows that $w \not\leq_T \gamma$. On the other hand $w^{-1} = \big(\!\big(a^{(0)}\ c^{(0)}\ b^{(d-1)}\big)\!\big)$, and

$$w\mathbf{v}' = \Big(\zeta_d v_{n-1}, v_1, \ldots, v_{c-1}, \ldots, \zeta_d^{d-1} v_{a-1}, \ldots, \zeta_d v_{b-1}, \ldots, v_{n-2}, \zeta_d^{d-1} v_n\Big)^{\mathsf{T}},$$

and it follows again that $w \not\leq_T \gamma$.

(x) Let $t_1 = \big(\!\big(a^{(0)}\ c^{(0)}\big)\!\big), t_2 = \big(\!\big(b^{(0)}\ c^{(d-1)}\big)\!\big)$, where $1 \leq a < b < c < n$. We have $w = \big(\!\big(a^{(0)}\ c^{(0)}\ b^{(1)}\big)\!\big)$, and

$$w^{-1}\mathbf{v}' = \Big(\zeta_d v_{n-1}, v_1, \ldots, v_{c-1}, \ldots, \zeta_d v_{a-1}, \ldots, \zeta_d^{d-1} v_{b-1}, \ldots, v_{n-2}, \zeta_d^{d-1} v_n\Big)^{\mathsf{T}},$$

and it follows that $w \not\leq_T \gamma$. On the other hand $w^{-1} = \big(\!\big(a^{(0)}\ b^{(1)}\ c^{(0)}\big)\!\big)$, and

$$w\mathbf{v}' = \Big(\zeta_d v_{n-1}, v_1, \ldots, \zeta_d^{d-1} v_{b-1}, \ldots, \zeta_d v_{c-1}, \ldots, v_{a-1}, \ldots, v_{n-2}, \zeta_d^{d-1} v_n\Big)^{\mathsf{T}},$$

and it follows again that $w^{-1} \not\leq_T \gamma$.

(xi) Let $t_1 = \big(\!\big(b^{(0)}\ c^{(0)}\big)\!\big), t_2 = \big(\!\big(a^{(0)}\ c^{(d-1)}\big)\!\big)$, where $1 \leq a < b < c < n$. We have $w = \big(\!\big(a^{(0)}\ b^{(d-1)}\ c^{(d-1)}\big)\!\big)$, and

$$w\mathbf{v}' = \Big(\zeta_d v_{n-1}, v_1, \ldots, \zeta_d v_{b-1}, \ldots, v_{c-1}, \ldots, \zeta_d^{d-1} v_{a-1}, \ldots, v_{n-2}, \zeta_d^{d-1} v_n\Big)^{\mathsf{T}},$$

and it follows that $w \leq_T \gamma$. Hence $w^{-1} = \big(\!\big(a^{(0)}\ c^{(d-1)}\ b^{(d-1)}\big)\!\big) \not\leq_T \gamma$. The reduced T-decompositions of w are

$$w = \big(\!\big(a^{(0)}\ b^{(d-1)}\big)\!\big)\big(\!\big(b^{(0)}\ c^{(0)}\big)\!\big) = \big(\!\big(b^{(0)}\ c^{(0)}\big)\!\big)\big(\!\big(a^{(0)}\ c^{(d-1)}\big)\!\big) = \big(\!\big(a^{(0)}\ c^{(d-1)}\big)\!\big)\big(\!\big(a^{(0)}\ b^{(d-1)}\big)\!\big),$$

and according to (4.13) only $w = \big(\!\big(b^{(0)}\ c^{(0)}\big)\!\big)\big(\!\big(a^{(0)}\ c^{(d-1)}\big)\!\big)$ is increasing.

(xii) Let $t_1 = \big(\!\big(b^{(0)}\ c^{(0)}\big)\!\big), t_2 = \big(\!\big(a^{(0)}\ b^{(d-1)}\big)\!\big)$, where $1 \leq a < b < c < n$. We have $w = \big(\!\big(a^{(0)}\ c^{(d-1)}\ b^{(d-1)}\big)\!\big)$, and this was already considered in (xi).

(xiii) Let $t_1 = \big(\!\big(a^{(0)}\ b^{(d-1)}\big)\!\big), t_2 = \big(\!\big(b^{(0)}\ c^{(d-1)}\big)\!\big)$, where $1 \leq a < b < c < n$. We have $w = \big(\!\big(a^{(0)}\ b^{(d-1)}\ c^{(d-2)}\big)\!\big)$, and

$$w^{-1}\mathbf{v}' = \Big(\zeta_d v_{n-1}, v_1, \ldots, \zeta_d v_{b-1}, \ldots, \zeta_d v_{c-1}, \ldots, \zeta_d^{d-2} v_{a-1}, \ldots, v_{n-2}, \zeta_d^{d-1} v_n\Big)^{\mathsf{T}},$$

and it follows that $w \not\leq_T \gamma$. On the other hand $w^{-1} = \big(\!\big(a^{(0)}\ c^{(d-2)}\ b^{(d-1)}\big)\!\big)$, and

$$w\mathbf{v}' = \Big(\zeta_d v_{n-1}, v_1, \ldots, \zeta_d^{d-2} v_{c-1}, \ldots, \zeta_d^{d-1} v_{a-1}, \ldots, \zeta_d^{d-1} v_{b-1}, \ldots, v_{n-2}, \zeta_d^{d-1} v_n\Big)^{\mathsf{T}},$$

and it follows again that $w^{-1} \not\leq_T \gamma$.

(xiv) Let $t_1 = \big(\!\big(a^{(0)}\ b^{(d-1)}\big)\!\big), t_2 = \big(\!\big(a^{(0)}\ c^{(d-1)}\big)\!\big)$, where $1 \leq a < b < c < n$. We have $w = \big(\!\big(a^{(0)}\ c^{(d-1)}\ b^{(d-1)}\big)\!\big)$, and this was already considered in (xii). On the other hand we have $w^{-1} = \big(\!\big(a^{(0)}\ b^{(d-1)}\ c^{(d-1)}\big)\!\big)$, and this was already considered in (xi).

(xv) Let $t_1 = \big(\!\big(a^{(0)}\ b^{(d-1)}\big)\!\big), t_2 = \big(\!\big(a^{(0)}\ n^{(s)}\big)\!\big)$, where $1 \leq a < b < n$ and $0 \leq s < d$. We have $w = \big(\!\big(a^{(0)}\ n^{(s)}\ b^{(d-1)}\big)\!\big)$, and

$$w^{-1}\mathbf{v}' = \Big(\zeta_d v_{n-1}, v_1, \ldots, \zeta_d^{d-1-s} v_n, \ldots, \zeta_d^{d-1} v_{a-1}, \ldots, v_{n-2}, \zeta_d^{s+1} v_{b-1}\Big)^{\mathsf{T}},$$

and it follows that $w \leq_T \gamma$. Hence $w^{-1} = \big(\!\big(a^{(0)}\ b^{(d-1)}\ n^{(s)}\big)\!\big) \not\leq_T \gamma$. The reduced T-decompositions of w are

$$w = \big(\!\big(a^{(0)}\ n^{(s)}\big)\!\big)\big(\!\big(b^{(0)}\ n^{(s+1)}\big)\!\big) = \big(\!\big(b^{(0)}\ n^{(s+1)}\big)\!\big)\big(\!\big(a^{(0)}\ b^{(d-1)}\big)\!\big) = \big(\!\big(a^{(0)}\ b^{(d-1)}\big)\!\big)\big(\!\big(a^{(0)}\ n^{(s)}\big)\!\big),$$

and according to (4.13) only $w = \big(\!\big(a^{(0)}\ n^{(s)}\big)\!\big)\big(\!\big(b^{(0)}\ n^{(s+1)}\big)\!\big)$ is increasing.

(xvi) Let $t_1 = \big(\!\big(a^{(0)}\ b^{(d-1)}\big)\!\big), t_2 = \big(\!\big(b^{(0)}\ n^{(s)}\big)\!\big)$, where $1 \leq a < b < c < n$. We have $w = \big(\!\big(a^{(0)}\ b^{(d-1)}\ n^{(s-1)}\big)\!\big)$, and this was already considered in (xv).

(xvii) Let $t_1 = \big(\!\big(a^{(0)}\ c^{(d-1)}\big)\!\big), t_2 = \big(\!\big(b^{(0)}\ c^{(d-1)}\big)\!\big)$, where $1 \leq a < b < c < n$. We have $w = \big(\!\big(a^{(0)}\ c^{(d-1)}\ b^{(0)}\big)\!\big)$, and this was already considered in (iv).

(xviii) Let $t_1 = \big((a^{(0)}\ n^{(s)})\big), t_2 = \big((a^{(0)}\ n^{(t)})\big)$, where $1 \leq a < n$ and $0 \leq s,t < d$ with $t \neq s$. We have $w = \big[a^{(0)}\big]_{t-s}\big[n^{(0)}\big]_{t-s}^{-1}$, and

$$w^{-1}\mathbf{v}' = \Big(\zeta_d v_{n-1}, v_1, \ldots, \zeta_d^{s-t} v_{a-1}, \ldots, v_{n-2}, \zeta_d^{t-1-s} v_n\Big)^{\mathsf{T}},$$

and it follows that $w \leq_T \gamma$ if and only if $t = s+1$. In this case the reduced T-decompositions of w are

$$\begin{aligned} w = \big((a^{(0)}\ n^{(s)})\big)\big((a^{(0)}\ n^{(s+1)})\big) &= \big((a^{(0)}\ n^{(s+1)})\big)\big((a^{(0)}\ n^{(s+2)})\big) \\ &= \big((a^{(0)}\ n^{(s+2)})\big)\big((a^{(0)}\ n^{(s+3)})\big) = \cdots = \big((a^{(0)}\ n^{(s-1)})\big)\big((a^{(0)}\ n^{(s)})\big). \end{aligned}$$

and according to (4.13) only $w = \big((a^{(0)}\ n^{(0)})\big)\big((a^{(0)}\ n^{(1)})\big)$ is increasing.

Thus the proof is complete. □

C.2. The Proof of Proposition 4.4.16

Proposition C.2.1

Let $w \leq_T \gamma$ such that the parabolic subgroup of $G(d,d,n)$, in which w is a Coxeter element, is isomorphic to $G(1,1,n')$ for some $n' \leq n$. Then, w is of one of the following three forms:

(i) $w = \big(((a{+}1)^{(0)}\ (a{+}2)^{(0)}\ \ldots\ b^{(0)})\big)$, *where* $1 \leq a < b < n$,

(ii) $w = \big((1^{(0)}\ 2^{(0)}\ \ldots\ a^{(0)}\ (b{+}1)^{(d-1)}\ (b{+}2)^{(d-1)}\ \ldots\ (n{-}1)^{(d-1)})\big)$, *where* $1 \leq a < b < n$, *or*

(iii) $w = \big((1^{(0)}\ 2^{(0)}\ \ldots\ a^{(0)}\ n^{(s-1)} (a{+}1)^{(d-1)}\ (a{+}2)^{(d-1)}\ \ldots\ (n{-}1)^{(d-1)})\big)$, *where* $1 \leq a < n$.

Moreover, in each of these cases there exists a unique rising reduced T-decomposition of w with respect to the restriction of $\prec_\gamma$ to the reflections in $T_\gamma \cap [\varepsilon, w]$.

Proof. The observation that w can only be of the forms (i)–(iii) is a straightforward computation using Proposition 4.4.8. For the second part of the proposition, we proceed by induction on $\ell_T(w)$. If $\ell_T(w) = 2$, then the claim follows from Lemma 4.4.15. Suppose that $\ell_T(w) = k$, and suppose that the claim is true for all suitable w' with $\ell_T(w') < k$.

(i) Let $w = \big(((a{+}1)^{(0)}\ (a{+}2)^{(0)}\ \ldots\ b^{(0)})\big)$, where $1 \leq a < b < n$. Consider the decomposition of w according to (4.11):

$$w = \big(((a{+}1)^{(0)}\ (a{+}2)^{(0)})\big)\big(((a{+}2)^{(0)}\ (a{+}3)^{(0)})\big)\cdots\big(((b{-}1)^{(0)}\ b^{(0)})\big).$$

We notice that this decomposition is rising with respect to (4.13), and the claim follows now analogously to the proof of Lemma 4.4.12.

(ii) Let $w = \big((1^{(0)}\ 2^{(0)}\ \ldots\ a^{(0)}\ (b{+}1)^{(d-1)}\ (b{+}2)^{(d-1)}\ \ldots\ (n{-}1)^{(d-1)})\big)$, where $1 \leq a < b < n$. Again consider the decomposition of w according to (4.11):

$$\begin{aligned} w = \big((1^{(0)}\ 2^{(0)})\big)\big((2^{(0)}\ 3^{(0)})\big)\cdots\big(((a{-}1)^{(0)}\ a^{(0)})\big)\big((a^{(0)}\ (b{+}1)^{(d-1)})\big) \\ \big(((b{+}1)^{(0)}\ (b{+}2)^{(0)})\big)\cdots\big(((n{-}2)^{(0)}\ (n{-}1)^{(0)})\big). \end{aligned}$$

We notice that this decomposition is not rising with respect to (4.13). However, repeated left-shifting yields

$$\begin{aligned} w = \big((1^{(0)}\ 2^{(0)})\big)\big((2^{(0)}\ 3^{(0)})\big)\cdots\big(((a{-}1)^{(0)}\ a^{(0)})\big) \\ \big(((b{+}1)^{(0)}\ (b{+}2)^{(0)})\big)\cdots\big(((n{-}2)^{(0)}\ (n{-}1)^{(0)})\big)\big((a^{(0)}\ (n{-}1)^{(d-1)})\big), \end{aligned} \tag{C.1}$$

and this decomposition is rising with respect to (4.13). We need to show that this is the only rising reduced T-decomposition of w. Suppose that $w = t_1 t_2 \cdots t_k$ is a rising reduced T-decomposition of w that is different from (C.1). Suppose that i is the maximal index where this decomposition differs from (C.1). If $i < k$, then $t_1 t_2 \cdots t_i$ is a product of at most two cycles of the form (i), and it follows that $t_1 t_2 \cdots t_i$ is rising only if t_j is the j-th factor in (C.1) for all $j \in \{1, 2, \ldots, i\}$, which is a contradiction. Now let $i = k$, and consider the word $w' = w t_k$. It follows by induction hypothesis that the product of the first $k-1$ factors in (C.1) is the unique rising reduced T-decomposition of w'. In view of Lemma 4.3.7 and Proposition 4.4.8 the reflection t_k can only be of one of the following four forms.
(iia) Let $t_k = \big(\big(a^{(0)}\ (n{-}1)^{(d-1)}\big)\big)$. Then, t_k is the k-th factor in (C.1), and we obtain a contradiction.
(iib) Let $t_k = \big(\big(a^{(0)}\ c^{(d-1)}\big)\big)$, where $b + 1 \leq c < n - 1$. We have

$$w' = \big(\big(1^{(0)}\ 2^{(0)}\ \ldots\ a^{(0)}\ (c{+}1)^{(d-1)}\ (c{+}2)^{(d-1)}\ \ldots\ (n{-}1)^{(d-1)}\big)\big)\big(\big((b{+}1)^{(0)}\ (b{+}2)^{(0)}\ \ldots\ c^{(0)}\big)\big).$$

Hence we can write $w' = w'_1 w'_2$, where w'_1 is again of type (ii) and w'_2 is of type (i). In particular $\ell_T(w'_1), \ell_T(w'_2) < k$, so by induction hypothesis w'_1 and w'_2 possess a unique rising decomposition, namely

$$\begin{aligned} w'_1 = \big(\big(1^{(0)}\ 2^{(0)}\big)\big)\big(\big(2^{(0)}\ 3^{(0)}\big)\big) \cdots \big(\big((a{-}1)^{(0)}\ a^{(0)}\big)\big) \\ \big(\big((c{+}1)^{(0)}\ (c{+}2)^{(0)}\big)\big) \cdots \big(\big((n{-}2)^{(0)}\ (n{-}1)^{(0)}\big)\big)\big(\big(a^{(0)}\ (n{-}1)^{(d-1)}\big)\big), \end{aligned}$$

and

$$w'_2 = \big(\big((b{+}1)^{(0)}\ (b{+}2)^{(0)}\big)\big)\big(\big((b{+}2)^{(0)}\ (b{+}3)^{(0)}\big)\big) \cdots \big(\big((c{-}1)^{(0)}\ c^{(0)}\big)\big).$$

Now we can quickly verify that

$$\begin{aligned} w' = \big(\big(1^{(0)}\ 2^{(0)}\big)\big) \cdots \big(\big((a{-}1)^{(0)}\ a^{(0)}\big)\big)\big(\big((b{+}1)^{(0)}\ (b{+}2)^{(0)}\big)\big) \cdots \big(\big((c{-}1)^{(0)}\ c^{(0)}\big)\big) \\ \big(\big((c{+}1)^{(0)}\ (c{+}2)^{(0)}\big)\big) \cdots \big(\big((n{-}2)^{(0)}\ (n{-}1)^{(0)}\big)\big)\big(\big(a^{(0)}\ (n{-}1)^{(d-1)}\big)\big), \end{aligned}$$

is the unique rising decomposition of w' and hence has to correspond to $t_1 t_2 \cdots t_{k-1}$. (Indeed, first concatenate the rising decompositions of w'_1 and w'_2, and observe that the resulting decomposition is not rising. Then, shift the first factor, say r, of the rising decomposition of w'_2 as far to the left as possible such that the resulting prefix, say $r_1 r_2 \cdots r_l$, is rising, where $r_1 r_2 \cdots r_{l-1}$ is a prefix of the rising decomposition of w'_1 and $r_l = r$. Then, observe that shifting r further to the left yields a non-rising prefix $r'_1 r'_2 \cdots r'_l$. Proceed analogously until you have reached the last factor of the rising decomposition of w'_2.) However, we have for instance $\big(\big(a^{(0)}\ (n{-}1^{(d-1)})\big)\big) \succ_\gamma \big(\big(a^{(0)}\ c^{(d-1)}\big)\big) = t_k$, which contradicts the assumption that $t_1 t_2 \cdots t_k$ is rising.
(iic) Let $t_k = \big(\big(c^{(0)}\ (c{+}1)^{(0)}\big)\big)$, where $b + 1 \leq c < n - 1$. We have

$$w' = \big(\big(1^{(0)}\ \ldots\ a^{(0)}\ (b{+}1)^{(d-1)}\ (b{+}2)^{(d-1)}\ \ldots\ c^{(d-1)}\ (c{+}2)^{(d-1)}\ (c{+}3)^{(d-1)}\ \ldots\ (n{-}1)^{(d-1)}\big)\big),$$

and $\ell_T(w') < k$. Moreover, w' is again of type (ii), so by induction hypothesis there exists a unique rising reduced T-decomposition of w', namely

$$\begin{aligned} w' = \big(\big(1^{(0)}\ 2^{(0)}\big)\big) \cdots \big(\big((a{-}1)^{(0)}\ a^{(0)}\big)\big)\big(\big((b{+}1)^{(0)}\ (b{+}2)^{(0)}\big)\big) \cdots \big(\big((c{-}1)^{(0)}\ c^{(0)}\big)\big)\big(\big(c^{(0)}\ c{+}2^{(0)}\big)\big) \\ \big(\big((c{+}2)^{(0)}\ (c{+}3)^{(0)}\big)\big) \cdots \big(\big((n{-}2)^{(0)}\ (n{-}1)^{(0)}\big)\big)\big(\big(a^{(0)}\ (n{-}1)^{(d-1)}\big)\big), \end{aligned}$$

and thus this decomposition has to correspond to $t_1 t_2 \cdots t_{k-1}$. However, we have for instance $\big(\big(a^{(0)}\ (n{-}1)^{(d-1)}\big)\big) \succ_\gamma \big(\big(c^{(0)}\ (c{+}1)^{(0)}\big)\big) = t_k$, which contradicts the assumption that $t_1 t_2 \cdots t_k$

is rising.
(iid) Let $t_k = \big(\!\big(c^{(0)}\ (c{+}1)^{(0)}\big)\!\big)$, where $1 \leq c < a$. We have

$$w' = \big(\!\big(1^{(0)}\ \ldots\ c^{(0)}\ (c+2)^{(0)}\ (c{+}3)^{(0)}\ \ldots\ a^{(0)}\ (b{+}1)^{(d-1)}\ (b{+}2)^{(d-1)}\ \ldots\ (n{-}1)^{(d-1)}\big)\!\big),$$

and $\ell_T(w') < k$. Moreover, w' is again of type (ii), so by induction hypothesis there exists a unique rising reduced T decomposition of w', namely

$$w' = \big(\!\big(1^{(0)}\ 2^{(0)}\big)\!\big) \cdots \big(\!\big(c^{(0)}\ (c{+}2)^{(0)}\big)\!\big)\big(\!\big((c{+}2)^{(0)}\ (c{+}3)^{(0)}\big)\!\big) \cdots \big(\!\big((a{-}1)^{(0)}\ a^{(0)}\big)\!\big)$$
$$\big(\!\big((b{+}1)^{(0)}\ (b{+}2)^{(0)}\big)\!\big)\big(\!\big((b{+}2)^{(0)}\ (b{+}3)^{(0)}\big)\!\big) \cdots \big(\!\big((n{-}2)^{(0)}\ (n{-}1)^{(0)}\big)\!\big)\big(\!\big(a^{(0)}\ (n{-}1)^{(d-1)}\big)\!\big),$$

and thus this decomposition has to correspond to $t_1t_2\cdots t_{k-1}$. However, we have for instance $\big(\!\big(a^{(0)}\ (n{-}1)^{(d-1)}\big)\!\big) \succ_\gamma \big(\!\big(c^{(0)}\ (c{+}1)^{(0)}\big)\!\big) = t_k$, which contradicts the assumption that $t_1t_2\cdots t_k$ is rising.
Hence the reduced T-decomposition of w in (C.1) is the unique rising reduced T-decomposition.

(iii) Let $w = \big(\!\big(1^{(0)}\ 2^{(0)}\ \ldots\ a^{(0)}\ n^{(s-1)} (a{+}1)^{(d-1)}\ (a{+}2)^{(d-1)}\ \ldots\ (n{-}1)^{(d-1)}\big)\!\big)$, where $1 \leq a < n$. Again consider the decomposition of w according to (4.11):

$$w = \big(\!\big(1^{(0)}\ 2^{(0)}\big)\!\big)\big(\!\big(2^{(0)}\ 3^{(0)}\big)\!\big) \cdots \big(\!\big((a{-}1)^{(0)}\ a^{(0)}\big)\!\big)\big(\!\big(a^{(0)}\ n^{(s-1)}\big)\!\big)\big(\!\big((a{+}1)^{(0)}\ n^{(s)}\big)\!\big)$$
$$\big(\!\big((a{+}1)^{(0)}\ (a{+}2)^{(0)}\big)\!\big)\big(\!\big((a{+}2)^{(0)}\ (a{+}3)^{(0)}\big)\!\big) \cdots \big(\!\big((n{-}2)^{(0)}\ (n{-}1)^{(0)}\big)\!\big).$$

We notice that this decomposition is not rising with respect to (4.13). However, repeated left-shifting yields

$$w = \big(\!\big(1^{(0)}\ 2^{(0)}\big)\!\big)\big(\!\big(2^{(0)}\ 3^{(0)}\big)\!\big) \cdots \big(\!\big((a{-}1)^{(0)}\ a^{(0)}\big)\!\big)\big(\!\big((a{+}1)^{(0)}\ (a{+}2)^{(0)}\big)\!\big)$$
$$\big(\!\big((a{+}2)^{(0)}\ (a{+}3)^{(0)}\big)\!\big) \cdots \big(\!\big((n{-}2)^{(0)}\ (n{-}1)^{(0)}\big)\!\big)\big(\!\big(a^{(0)}\ n^{(s-1)}\big)\!\big)\big(\!\big((n{-}1)^{(0)}\ n^{(s)}\big)\!\big), \tag{C.2}$$

and this decomposition is rising with respect to (4.13). We need to show that this is the only rising reduced T-decomposition of w. Again we suppose that $w = t_1t_2\cdots t_k$ is a rising reduced T-decomposition of w that is different from (C.2), and analogously to (ii) it suffices to investigate $w' = wt_k$. In view of Lemma 4.3.7 and Proposition 4.4.8 the reflection t_k can only be of one of the following five forms.
(iiia) Let $t_k = \big(\!\big((n{-}1)^{(0)}\ n^{(s)}\big)\!\big)$. Then, t_k is the k-th factor in (C.2), and we obtain a contradiction.
(iiib) Let $t_k = \big(\!\big(c^{(0)}\ n^{(s)}\big)\!\big)$, where $a+1 \leq c < n-1$. We have

$$w' = \big(\!\big(1^{(0)}\ 2^{(0)}\ \ldots\ a^{(0)}\ n^{(s-1)}\ (c{+}1)^{(d-1)}\ (c{+}2)^{(d-1)}\ \ldots\ (n{-}1)^{(d-1)}\big)\!\big)$$
$$\big(\!\big((a{+}1)^{(0)}\ (a{+}2)^{(0)}\ \ldots\ c^{(0)}\big)\!\big).$$

Hence we can write $w' = w_1'w_2'$, where w_1' is again of type (iii) and w_2' is of type (i). In particular $\ell_T(w_1'), \ell_T(w_2') < k$, so by induction hypothesis we can find a unique rising decomposition of w' analogously to (iib), namely

$$w' = \big(\!\big(1^{(0)}\ 2^{(0)}\big)\!\big)\big(\!\big(2^{(0)}\ 3^{(0)}\big)\!\big) \cdots \big(\!\big((a{-}1)^{(0)}\ a^{(0)}\big)\!\big)\big(\!\big((a{+}1)^{(0)}\ (a{+}2)^{(0)}\big)\!\big) \cdots \big(\!\big((c{-}1)^{(0)}\ c^{(0)}\big)\!\big)$$
$$\big(\!\big((c{+}1)^{(0)}\ (c{+}2)^{(0)}\big)\!\big) \cdots \big(\!\big((n{-}2)^{(0)}\ (n{-}1)^{(0)}\big)\!\big)\big(\!\big(a^{(0)}\ n^{(s-1)}\big)\!\big)\big(\!\big((n{-}1)^{(0)}\ n^{(s)}\big)\!\big),$$

and thus this decomposition has to correspond to $t_1t_2\cdots t_{k-1}$. However, we have for instance $\big(\!\big((n{-}1)^{(0)}\ n^{(s)}\big)\!\big) \succ_\gamma \big(\!\big(c^{(0)}\ n^{(s)}\big)\!\big) = t_k$, which implies that there exists no rising reduced T-decomposition of w in this case.

(iiic) Let $t_k = (\!(c^{(0)}\ n^{(s-1)})\!)$, where $a \leq c < n-1$. We have

$$w' = (\!(1^{(0)}\ 2^{(0)}\ \ldots\ a^{(0)}\ n^{(s-1)}\ (c{+}1)^{(0)}\ (c{+}2)^{(0)}\ \ldots\ (n{-}1)^{(0)})\!)\\ (\!((a{+}1)^{(0)}\ (a{+}2)^{(0)}\ \ldots\ c^{(0)})\!).$$

Hence we can write $w' = w'_1 w'_2$, where w'_1 is again of type (iii) and w'_2 is of type (i). In particular $\ell_T(w'_1), \ell_T(w'_2) < k$, so by induction hypothesis we can find a unique rising decomposition of w' analogously to (iiib), namely

$$w' = (\!(1^{(0)}\ 2^{(0)})\!)(\!(2^{(0)}\ 3^{(0)})\!)\cdots(\!((a{-}1)^{(0)}\ a^{(0)})\!)(\!((a{+}1)^{(0)}\ (a{+}2)^{(0)})\!)\cdots(\!((c{-}1)^{(0)}\ c^{(0)})\!)\\ (\!((c{+}1)^{(0)}\ (c{+}2)^{(0)})\!)\cdots(\!((n{-}2)^{(0)}\ (n{-}1)^{(0)})\!)(\!(c^{(0)}\ n^{(s-1)})\!)(\!((n{-}1)^{(0)}\ n^{(s-1)})\!),$$

and thus this decomposition has to correspond to $t_1 t_2 \cdots t_{k-1}$. However, we have for instance $(\!((n{-}1)^{(0)}\ n^{(s-1)})\!) \succ_\gamma (\!(c^{(0)}\ n^{(s-1)})\!) = t_k$, which contradicts the assumption that $t_1 t_2 \cdots t_k$ is rising.

(iiid) Let $t_k = (\!(c^{(0)}\ (c{+}1)^{(0)})\!)$, where $a+1 \leq c < n-1$. We have

$$w' = (\!(1^{(0)}\ \ldots\ a^{(0)}\ n^{(s-1)}\ (a{+}1)^{(d-1)}\ \ldots\ c^{(d-1)}\ (c{+}2)^{(d-1)}\ (c{+}3)^{(d-1)}\ \ldots\ (n{-}1)^{(d-1)})\!),$$

and $\ell_T(w') < k$. Moreover, w' is again of type (iii), so by induction hypothesis there exists a unique rising reduced T decomposition of w', namely

$$w' = (\!(1^{(0)}\ 2^{(0)})\!)\cdots(\!((a{-}1)^{(0)}\ a^{(0)})\!)(\!((a{+}1)^{(0)}\ (a{+}2)^{(0)})\!)\cdots(\!((c{-}1)^{(0)}\ c^{(0)})\!)(\!(c^{(0)}\ c{+}2^{(0)})\!)\\ (\!((c{+}2)^{(0)}\ (c{+}3)^{(0)})\!)\cdots(\!((n{-}2)^{(0)}\ (n{-}1)^{(0)})\!)(\!(a^{(0)}\ (n{-}1)^{(s-1)})\!)(\!((n{-}1)^{(0)}\ n^{(s)})\!),$$

and thus this decomposition has to correspond to $t_1 t_2 \cdots t_{k-1}$. However, we have for instance $(\!((n{-}1)^{(0)}\ n^{(s)})\!) \succ_\gamma (\!(c^{(0)}\ (c{+}1)^{(0)})\!) = t_k$, which contradicts the assumption that $t_1 t_2 \cdots t_k$ is rising.

(iiie) Let $t_k = (\!(c^{(0)}\ (c{+}1)^{(0)})\!)$, where $1 \leq c < a$. We have

$$w' = (\!(1^{(0)}\ \ldots\ c^{(0)}\ (c+2)^{(0)}\ (c{+}3)^{(0)}\ \ldots\ a^{(0)}\ n^{(s-1)}\ (a{+}1)^{(d-1)}\ \ldots\ (n{-}1)^{(d-1)})\!),$$

and $\ell_T(w') < k$. Moreover, w' is again of type (iii), so by induction hypothesis there exists a unique rising reduced T decomposition of w', namely

$$w' = (\!(1^{(0)}\ 2^{(0)})\!)\cdots(\!((c{-}1)^{(0)}\ c^{(0)})\!)(\!(c^{(0)}\ (c{+}2)^{(0)})\!)(\!((c{+}2)^{(0)}\ (c{+}3)^{(0)})\!)\cdots(\!((a{-}1)^{(0)}\ a^{(0)})\!)\\ (\!((a{+}1)^{(0)}\ (a{+}2)^{(0)})\!)\cdots(\!((n{-}2)^{(0)}\ (n{-}1)^{(0)})\!)(\!(a^{(0)}\ n^{(s-1)})\!)(\!((n{-}1)^{(0)}\ n^{(s)})\!),$$

and thus this decomposition has to correspond to $t_1 t_2 \cdots t_{k-1}$. However, we have for instance $(\!((n{-}1)^{(0)}\ n^{(s)})\!) \succ_\gamma (\!(c^{(0)}\ (c{+}1)^{(0)})\!) = t_k$, which contradicts the assumption that $t_1 t_2 \cdots t_k$ is rising.

Hence the reduced T-decomposition of w in (C.2) is the unique rising reduced T-decomposition, and the proof is complete. □

C.3. The Proof of Corollary 4.4.17

COROLLARY C.3.1

Let $w \leq_T \gamma$ such that the parabolic subgroup W of $G(d,d,n)$, in which w is a Coxeter element, is reducible, and hence $W = W_1 \times W_2 \times \cdots \times W_l$ for some l. If for each $i \in \{1,2,\ldots,l\}$, the group W_i is isomorphic to $G(1,1,n_i)$ for $n_i \leq n$, then there exists a unique rising reduced T-decomposition of w with respect to the restriction of $\prec_\gamma$ to the reflections in $T_\gamma \cap [\varepsilon, w]$.

PROOF. First suppose that $l = 2$. In particular, we can write $w = w_1 w_2$, where w_1 and w_2 commute. In view of Proposition 4.4.16, each of w_1 and w_2 can be of three possible forms. Since they commute it suffices to consider the following cases:

(i) Let $w_1 = ((a^{(0)} \; \dots \; b^{(0)}))$ and $w_2 = ((c^{(0)} \; \dots \; e^{(0)}))$, where $a < b < e + 1 < d$. Proposition 4.4.16 implies that each of w_1 and w_2 has a unique rising reduced T-decomposition, namely

$$w_1 = ((a^{(0)} \; (a{+}1)^{(0)}))\,(((a{+}1)^{(0)} \; (a{+}2)^{(0)})) \cdots (((b{-}1)^{(0)} \; b^{(0)})), \quad \text{and}$$
$$w_2 = ((c^{(0)} \; (c{+}1)^{(0)}))\,(((c{+}1)^{(0)} \; (c{+}2)^{(0)})) \cdots (((e{-}1)^{(0)} \; e^{(0)})),$$

and the concatenation $w_1 w_2$ is clearly the unique rising reduced T-decomposition of w.

(ii) Let $w_1 = ((a^{(0)} \; \dots \; b^{(0)}))$ and $w_2 = ((c^{(0)} \; \dots \; e^{(d-1)} \; \dots \; (n{-}1)^{(d-1)}))$, where $a < b < c + 1 < e$. Again Proposition 4.4.16 implies that each of w_1 and w_2 has a unique rising reduced T-decomposition, namely

$$\begin{aligned} w_1 &= ((a^{(0)} \; (a{+}1)^{(0)}))\,(((a{+}1)^{(0)} \; (a{+}2)^{(0)})) \cdots (((b{-}1)^{(0)} \; b^{(0)})), \quad \text{and} \\ w_2 &= ((c^{(0)} \; (c{+}1)^{(0)}))\,(((c{+}1)^{(0)} \; (c{+}2)^{(0)})) \cdots (((e{-}2)^{(0)} \; (e{-}1)^{(0)})) \\ &\qquad ((e^{(0)} \; (e{+}1)^{(0)})) \cdots (((n{-}2)^{(0)} \; (n{-}1)^{(0)}))\,(((e{-}1)^{(0)} \; (n{-}1)^{(d-1)})), \end{aligned}$$

and the concatenation $w_1 w_2$ is clearly the unique rising reduced T-decomposition of w.

(iii) Let $w_1 = ((a^{(0)} \; \dots \; b^{(0)}))$ and $w_2 = ((c^{(0)} \; \dots \; e^{(d-1)} \; n^{(s-1)} \; (e{+}1)^{(d-1)} \; \dots \; (n{-}1)^{(d-1)}))$, where $a < b < c + 1 < e$. Again Proposition 4.4.16 implies that each of w_1 and w_2 has a unique rising reduced T-decomposition, namely

$$\begin{aligned} w_1 &= ((a^{(0)} \; (a{+}1)^{(0)}))\,(((a{+}1)^{(0)} \; (a{+}2)^{(0)})) \cdots (((b{-}1)^{(0)} \; b^{(0)})), \quad \text{and} \\ w_2 &= ((c^{(0)} \; (c{+}1)^{(0)}))\,(((c{+}1)^{(0)} \; (c{+}2)^{(0)})) \cdots (((e{-}2)^{(0)} \; (e{-}1)^{(0)})) \cdots (((n{-}2)^{(0)} \; (n{-}1)^{(0)})) \\ &\qquad (((e{-}1)^{(0)} \; (n{-}1)^{(d-1)}))\,((e^{(0)} \; n^{(s)}))\,(((n{-}1)^{(0)} \; n^{(s)})), \end{aligned}$$

and the concatenation $w_1 w_2$ is clearly the unique rising reduced T-decomposition of w.

(iv) Let $w_1 = ((a^{(0)} \; \dots \; b^{(d-1)} \; \dots \; (c{-}1)^{(d-1)}))$ and $w_2 = ((c^{(0)} \; \dots \; e^{(d-1)} \; \dots \; (n{-}1)^{(d-1)}))$, where $a < b < c + 1 < e$. Again Proposition 4.4.16 implies that each of w_1 and w_2 has a unique rising reduced T-decomposition, namely

$$\begin{aligned} w_1 &= ((a^{(0)} \; (a{+}1)^{(0)}))\,(((a{+}1)^{(0)} \; (a{+}2)^{(0)})) \cdots (((b{-}2)^{(0)} \; (b{-}1)^{(0)})) \\ &\qquad ((b^{(0)} \; (b{+}1)^{(0)})) \cdots (((c{-}2)^{(0)} \; (c{-}1)^{(0)}))\,(((b{-}1)^{(0)} \; (c{-}1)^{(d-1)})), \quad \text{and} \\ w_2 &= ((c^{(0)} \; (c{+}1)^{(0)}))\,(((c{+}1)^{(0)} \; (c{+}2)^{(0)})) \cdots (((e{-}2)^{(0)} \; (e{-}1)^{(0)})) \\ &\qquad ((e^{(0)} \; (e{+}1)^{(0)})) \cdots (((n{-}2)^{(0)} \; (n{-}1)^{(0)}))\,(((e{-}1)^{(0)} \; (n{-}1)^{(d-1)})). \end{aligned}$$

Since w_1 and w_2 commute, it is easy to see that there is a unique rising reduced T-decomposition of w, namely

$$\begin{aligned} w &= ((a^{(0)} \; (a{+}1)^{(0)})) \cdots (((b{-}2)^{(0)} \; (b{-}1)^{(0)}))\,((b^{(0)} \; (b{+}1)^{(0)})) \cdots (((c{-}2)^{(0)} \; (c{-}1)^{(0)})) \\ &\qquad ((c^{(0)} \; (c{+}1)^{(0)})) \cdots (((e{-}2)^{(0)} \; (e{-}1)^{(0)}))\,((e^{(0)} \; (e{+}1)^{(0)})) \cdots (((n{-}2)^{(0)} \; (n{-}1)^{(0)})) \\ &\qquad (((b{-}1)^{(0)} \; (c{-}1)^{(d-1)}))\,(((e{-}1)^{(0)} \; (n{-}1)^{(d-1)})). \end{aligned}$$

(v) Let $w_1 = ((a^{(0)} \; \dots \; b^{(d-1)} \; \dots \; (c{-}1)^{(d-1)}))$ and $w_2 = ((c^{(0)} \; \dots \; e^{(d-1)} \; n^{(s-1)} \; (e{+}1)^{(d-1)} \; \dots \; (n{-}1)^{(d-1)}))$, where $a < b < c + 1 < e$. Again Proposition 4.4.16 implies that each of w_1 and w_2 has a unique rising reduced T-decomposition, namely

$$\begin{aligned} w_1 = {} & \big(\!\big(a^{(0)}\ (a{+}1)^{(0)}\big)\!\big)\big(\!\big((a{+}1)^{(0)}\ (a{+}2)^{(0)}\big)\!\big)\cdots\big(\!\big((b{-}2)^{(0)}\ (b{-}1)^{(0)}\big)\!\big)\\ & \big(\!\big(b^{(0)}\ (b{+}1)^{(0)}\big)\!\big)\cdots\big(\!\big((c{-}2)^{(0)}\ (c{-}1)^{(0)}\big)\!\big)\big(\!\big((b{-}1)^{(0)}\ (c{-}1)^{(d-1)}\big)\!\big), \quad \text{and}\\ w_2 = {} & \big(\!\big(c^{(0)}\ (c{+}1)^{(0)}\big)\!\big)\big(\!\big((c{+}1)^{(0)}\ (c{+}2)^{(0)}\big)\!\big)\cdots\big(\!\big((e{-}2)^{(0)}\ (e{-}1)^{(0)}\big)\!\big)\cdots\big(\!\big((n{-}2)^{(0)}\ (n{-}1)^{(0)}\big)\!\big)\\ & \big(\!\big((e{-}1)^{(0)}\ (n{-}1)^{(d-1)}\big)\!\big)\big(\!\big(e^{(0)}\ n^{(s)}\big)\!\big)\big(\!\big((n{-}1)^{(0)}\ n^{(s)}\big)\!\big). \end{aligned}$$

Since w_1 and w_2 commute, it is easy to see that there is a unique rising reduced T-decomposition of w, namely

$$\begin{aligned} w = {} & \big(\!\big(a^{(0)}\ (a{+}1)^{(0)}\big)\!\big)\cdots\big(\!\big((b{-}2)^{(0)}\ (b{-}1)^{(0)}\big)\!\big)\big(\!\big(b^{(0)}\ (b{+}1)^{(0)}\big)\!\big)\cdots\big(\!\big((c{-}2)^{(0)}\ (c{-}1)^{(0)}\big)\!\big)\\ & \big(\!\big(c^{(0)}\ (c{+}1)^{(0)}\big)\!\big)\cdots\big(\!\big((e{-}2)^{(0)}\ (e{-}1)^{(0)}\big)\!\big)\cdots\big(\!\big((n{-}2)^{(0)}\ (n{-}1)^{(0)}\big)\!\big)\\ & \big(\!\big((b{-}1)^{(0)}\ (c{-}1)^{(d-1)}\big)\!\big)\big(\!\big((e{-}1)^{(0)}\ (n{-}1)^{(d-1)}\big)\!\big)\big(\!\big(e^{(0)}\ n^{(s)}\big)\!\big)\big(\!\big((n{-}1)^{(0)}\ n^{(s)}\big)\!\big). \end{aligned}$$

The case that both w_1 and w_2 are of type (iii) in Proposition 4.4.16 cannot occur, since in this case w_1 and w_2 would not commute. The proof for $l > 2$ works analogously. $\square$

Bibliography

[1] Drew Armstrong, *Generalized Noncrossing Partitions and Combinatorics of Coxeter Groups*, Memoirs of the American Mathematical Society **202** (2009).

[2] Drew Armstrong and Christian Krattenthaler, *Euler Characteristic of the Truncated Order Complex of Generalized Noncrossing Partitions*, The Electronic Journal of Combinatorics **16** (2009).

[3] Drew Armstrong, Brendan Rhoades, and Nathan Williams, *Rational Associahedra and Noncrossing Partitions*, The Electronic Journal of Combinatorics **20** (2013).

[4] Christos A. Athanasiadis, *Generalized Catalan Numbers, Weyl Groups and Arrangements of Hyperplanes*, Bulletin of the London Mathematical Society **36** (2004), 294–302.

[5] Christos A. Athanasiadis, *On a Refinement of the Generalized Catalan Numbers for Weyl Groups*, Transactions of the American Mathematical Society **357** (2005), 179–196.

[6] Christos A. Athanasiadis, Thomas Brady, and Colum Watt, *Shellability of Noncrossing Partition Lattices*, Proceedings of the American Mathematical Society **135** (2007), 939–949.

[7] Christos A. Athanasiadis and Victor Reiner, *Noncrossing Partitions for the Group D_n*, SIAM Journal on Discrete Mathematics **18** (2004), 397–417.

[8] Jean-Christophe Aval, *Multivariate Fuss-Catalan Numbers*, Discrete Mathematics **308** (2008), 4660–4669.

[9] Bernhard Banaschewski, *Hüllensysteme und Erweiterungen von Quasi-Ordnungen*, Zeitschrift für mathematische Logik und Grundlagen der Mathematik **2** (1956), 117–130.

[10] Elena Barcucci, Antonio Bernini, Luca Ferrari, and Maddalena Poneti, *A Distributive Lattice Structure Connecting Dyck Paths, Noncrossing Partitions and 312-Avoiding Permutations*, Order **22** (2005), 311–328.

[11] François Bergeron and Louis-François Préville-Ratelle, *Higher Trivariate Diagonal Harmonics via Generalized Tamari Posets*, Journal of Combinatorics **3** (2012), 317–341.

[12] David Bessis, *The Dual Braid Monoid*, Annales Scientifiques de l'École Normale Supérieure **36** (2003), 647–683.

[13] David Bessis, *Finite Complex Reflection Arrangements are $K(\pi, 1)$* (2007), available at `arXiv:0610.777`.

[14] David Bessis and Ruth Corran, *Non-crossing Partitions of Type (e, e, r)*, Advances in Mathematics **202** (2006), 1–49.

[15] David Bessis, François Digne, and Jean Michel, *Springer Theory in Braid Groups and the Birman-Ko-Lee Monoid*, Pacific Journal of Mathematics **205** (2002), 287–309.

[16] David Bessis and Victor Reiner, *Cyclic Sieving of Noncrossing Partitions for Complex Reflection Groups*, Annals of Combinatorics **15** (2011), 197–222.

[17] Philippe Biane, *Some Properties of Crossings and Partitions*, Discrete Mathematics **175** (1997), 41–53.

[18] Philippe Biane, Frederick Goodman, and Alexandru Nica, *Non-Crossing Cumulants of Type B*, Transactions of the American Mathematical Society **355** (2003), 2263–2303.

[19] Joan Birman, Ki Hyoung Ko, and Sang Jin Lee, *A New Approach to the Word and Conjugacy Problems in the Braid Groups*, Advances in Mathematics **139** (1998), 322–353.

[20] Michael T. L. Bizley, *Derivation of a New Formula for the Number of Minimal Lattice Paths from $(0, 0)$ to (km, kn) Having just t Contacts with the Line $my = nx$ and Having no Points above this Line; and a Proof of Grossman's Formula for the Number of Paths which May Touch but Do not Rise above this Line*, Journal for the Institute of Actuaries **80** (1954), 55–62.

[21] Anders Björner, *Shellable and Cohen-Macaulay Partially Ordered Sets*, Transactions of the American Mathematical Society **260** (1980), 159–183.

[22] Anders Björner, *Topological Methods*, Handbook of Combinatorics, vol. 2, MIT Press, Cambridge, MA, 1996.

[23] Anders Björner and Francesco Brenti, *Combinatorics of Coxeter Groups*, Springer, New York, 2005.

[24] Anders Björner and Michelle L. Wachs, *On Lexicographically Shellable Posets*, Transactions of the American Mathematical Society **277** (1983), 323–341.

[25] Anders Björner and Michelle L. Wachs, *Shellable and Nonpure Complexes and Posets I*, Transactions of the American Mathematical Society **348** (1996), 1299–1327.

[26] Anders Björner and Michelle L. Wachs, *Shellable and Nonpure Complexes and Posets II*, Transactions of the American Mathematical Society **349** (1997), 3945–3975.

[27] Andreas Blass and Bruce E. Sagan, *Möbius Functions of Lattices*, Advances in Mathematics **127** (1997), 94–123.

[28] Daniel Borchmann, *conexp-clj: An Extensive Tool for Computations in Formal Concept Analysis.* https://github.com/exot/conexp-clj/.

[29] Raoul Bott and Clifford H. Taubes, *On the Self-Linking of Knots*, Journal of Mathematical Physics **35** (1994), 5247–5287.

[30] Mireille Bousquet-Mélou, Éric Fusy, and Louis-François Préville-Ratelle, *The Number of Intervals in the m-Tamari Lattices*, The Electronic Journal of Combinatorics **18** (2011).

[31] Thomas Brady, *A Partial Order on the Symmetric Group and new $K(\pi,1)$'s for the Braid Groups*, Advances in Mathematics **161** (2001), 20–40.

[32] Thomas Brady and Colum Watt, *A Partial Order on the Orthogonal Group*, Communications in Algebra **30** (2002), 3749–3754.

[33] Thomas Brady and Colum Watt, *$K(\pi,1)$'s for Artin Groups of Finite Type*, Geometriae Dedicata **94** (2002), 225–250.

[34] Thomas Brady and Colum Watt, *Non-Crossing Partition Lattices in Finite Real Reflection Groups*, Transactions of the American Mathematical Society **360** (2008), 1983–2005.

[35] Michel Broué, Gunter Malle, and Raphaël Rouquier, *Complex Reflection Groups, Braid Groups, Hecke Algebras*, Journal für die reine und angewandte Mathematik **500** (1998), 127–190.

[36] Nathalie Caspard and Claude Le Conte de Poly-Barbut, *Tamari Lattices are Bounded—A new Proof*, Technical Report TR-2004-03, Laboratory of Algorithms, Complexity and Logic, University of Paris XII, 2004.

[37] Nathalie Caspard, Claude Le Conte de Poly-Barbut, and Michel Morvan, *Cayley Lattices of Finite Coxeter Groups are Bounded*, Advances in Applied Mathematics **33** (2004), 71–94.

[38] Guillaume Chapuy and Christian Stump, *Counting Factorizations of Coxeter Elements into Products of Reflections* (2012), available at arXiv:1211.2789.

[39] Claude Chevalley, *Invariants of Finite Groups Generated by Reflections*, American Journal of Mathematics **77** (1955), 778–782.

[40] Harold S. M. Coxeter, *The Complete Enumeration of Finite Groups of the Form $R_i^2 = (R_iR_j)^{k_{ij}} = 1$*, Journal of the London Mathematical Society **10** (1935), 21–25.

[41] Brian A. Davey, Werner Poguntke, and Ivan Rival, *A Characterization of Semi-Distributivity*, Algebra Universalis **5** (1975), 72–75.

[42] Brian A. Davey and Hilary A. Priestley, *Introduction to Lattices and Order*, Cambridge University Press, Cambridge, 2002.

[43] Alan Day, *Characterizations of Finite Lattices that are Bounded-Homomorphic Images or Sublattices of Free Lattices*, Canadian Journal of Mathematics **31** (1979), 69–78.

[44] Dragomir Ž. Djoković, *On Conjugacy Classes of Elements of Finite Order in Compact or Complex Semisimple Lie Groups*, Proceedings of the American Mathematical Society **80** (1980), 181–184.

[45] Philippe Duchon, *On the Enumeration and Generation of Generalized Dyck Words*, Discrete Mathematics **225** (2000), 121–135.

[46] Eugene B. Dynkin, *The Structure of Semi-Simple Algebras*, Uspekhi Matematicheskikh Nauk **4** (1947), 59–127.

[47] Paul H. Edelman, *Chain Enumeration and Non-Crossing Partitions*, Discrete Mathematics **31** (1980), 171–180.

[48] Luca Ferrari, *Unimodality and Dyck Paths*, Journal of Combinatorial Mathematics and Combinatorial Computing (2013). To appear.

[49] Luca Ferrari and Emanuele Munarini, *Lattices of Paths: Representation Theory and Valuations*, Journal of Combinatorics **2** (2011), 265–291.

[50] Luca Ferrari and Emanuele Munarini, *Enumeration of Edges in Some Lattices of Paths* (2012), available at arXiv:1203.6792.

[51] Luca Ferrari and Emanuele Munarini, *Enumeration of Saturated Chains in Dyck Lattices* (2012), available at arXiv:1203.6807.

[52] Luca Ferrari and Renzo Pinzani, *Lattices of Lattice Paths*, Journal of Statistical Planning and Inference **135** (2005), 77–92.

[53] Sergey Fomin and Nathan Reading, *Generalized Cluster Complexes and Coxeter Combinatorics*, International Mathematics Research Notices **44** (2005), 2709–2757.

[54] Sergey Fomin and Andrei Zelevinsky, *Y-Systems and Generalized Associahedra*, Annals of Mathematics **158** (2003), 977–1018.

[55] Ralph Freese, Jaroslav Ježek, and James B. Nation, *Free Lattices*, American Mathematical Society, Providence, 1995.

[56] Haya Friedman and Dov Tamari, *Problémes d'Associativité: Une Structure de Treillis Finis Induite par une Loi Demi-Associative*, Journal of Combinatorial Theory **2** (1967), 215–242.

[57] Bernhard Ganter and Rudolf Wille, *Formal Concept Analysis: Mathematical Foundations*, Springer, Heidelberg, 1999.

[58] Winfried Geyer, *On Tamari Lattices*, Discrete Mathematics **133** (1994), 99–122.

[59] James Haglund, Mark Haiman, Nicholas Loehr, Jeffrey B. Remmel, and Alexander Ulyanov, *A Combinatorial Formula for the Character of the Diagonal Coinvariants*, Duke Mathematical Journal **126** (2005), 195–232.

[60] Mark Haiman, *Conjectures on the Quotient Ring by Diagonal Invariants*, Journal of Algebraic Combinatorics **3** (1994), 17–76.

[61] Philip Hall, *The Eulerian Functions of a Group*, The Quarterly Journal of Mathematics **7** (1936), 134–151.

[62] Samuel Huang and Dov Tamari, *Problems of Associativity: a Simple Proof for the Lattice Property of Systems Ordered by a Semi-Associative Law*, Journal of Combinatorial Theory (Series A) **13** (1972), 7–13.

[63] James E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge University Press, Cambridge, 1990.

[64] Colin Ingalls and Hugh Thomas, *Noncrossing Partitions and Representations of Quivers*, Composition Mathematica **145** (2009), 1533–1562.

[65] Přemsyl Jedlička, *A Combinatorial Construction of the Weak Order of a Coxeter Group*, Communications in Algebra **33** (2005), 1447–1460.

[66] Myrto Kallipoliti and Henri Mühle, *On the Topology of the Cambrian Semilattices*, The Electronic Journal of Combinatorics **20** (2013).

[67] Myrto Kallipoliti and Henri Mühle, *The m-Cover Posets and the Strip-Decomposition of m-Dyck Paths* (2013), available at `arXiv:1312.2520`.

[68] Jang Soo Kim, Karola Mészáros, Greta Panova, and David B. Wilson, *Dyck Tilings, Increasing Trees, Descents and Inversions*, Journal of Combinatorial Theory (Series A) **122** (2014), 9–27.

[69] Christian Krattenthaler, *Counting Lattice Paths with a Linear Boundary*, Sitzungsberichte der ÖAW, Mathematisch-naturwissenschaftliche Klasse **198** (1989), 87–107.

[70] Germain Kreweras, *Sur les partitions non croisées d'un cycle*, Discrete Mathematics **1** (1972), 333–350.

[71] Gustav I. Lehrer and Jean Michel, *Invariant Theory and Eigenspaces for Unitary Reflection Groups*, Comptes Rendus de l'Académie des Sciences Paris **336** (2003), 795–800.

[72] Gustav I. Lehrer and Tonny A. Springer, *Reflection Subquotients of Unitary Reflection Groups*, Canadian Journal of Mathematics **51** (1999), 1175–1193.

[73] Gustav I. Lehrer and Donald E. Taylor, *Unitary Reflection Groups*, Cambridge University Press, Cambridge, 2009.

[74] Shu-Chung Liu, *Left-Modular Elements and Edge-Labellings*, Ph.D. Thesis, Michigan State University, 1999.

[75] Shu-Chung Liu and Bruce E. Sagan, *Left-Modular Elements of Lattices*, Journal of Combinatorial Theory (Series A) **91** (2000), 369–385.

[76] George Markowsky, *Primes, Irreducibles and Extremal Lattices*, Order **9** (1992), 265–290.

[77] Jon McCammond, *Noncrossing Partitions in Surprising Locations*, American Mathematical Monthly **113** (2006), 598–610.

[78] Ralph McKenzie, *Equational Bases and Nonmodular Lattice Variety*, Transactions of the American Mathematical Society **174** (1975), 1–43.

[79] Peter McNamara and Hugh Thomas, *Poset Edge-Labellings and Left Modularity*, European Journal of Combinatorics **27** (2006), 101–113.

[80] Jean Michel, *gap3-jm: A GAP3 Distribution*. `http://www.math.jussieu.fr/~jmichel/gap3/`.

[81] Henri Mühle, *Lins: A Labeling Inspector for Well-Generated Complex Reflection Groups*. `http://homepage.univie.ac.at/henri.muehle/misc.php`.

[82] Henri Mühle, *EL-Shellability and Noncrossing Partitions Associated to Well-Generated Complex Reflection Groups* (2011), available at `arXiv:1111.7172`.

[83] Henri Mühle, *On the EL-Shellability of the m-Tamari Lattices* (2012), available at `arXiv:1201.2020`.

[84] Henri Mühle, *A Heyting Algebra on Dyck Paths of Type A and B* (2013), available at `arXiv:1312.0551`.

[85] Henri Mühle, *Structural Properties of the Cambrian Semilattices – Consequences of Semidistributivity* (2013), available at `arXiv:1312.4449`.

[86] Folkert Müller-Hoissen, Jean Marcel Pallo, and Jim Stasheff (Eds.), *Associahedra, Tamari Lattices and Related Structures*, Birkhäuser, Basel, 2012.

[87] Peter Orlik and Louis Solomon, *Unitary Reflection Groups and Cohomology*, Inventiones Mathematicae **59** (1980), 77–94.

[88] Peter Orlik and Louis Solomon, *Discriminants in the Invariant Theory of Reflection Groups*, Nagoya Mathematical Journal **109** (1988), 23–45.

[89] Jean Marcel Pallo, *An Algorithm to Compute the Möbius Function of the Rotation Lattice of Binary Trees*, Informatique Théorique et Applications **27** (1993), 341–348.

[90] Vincent Pilaud and Christian Stump, *Brick Polytopes of Spherical Subword Complexes: A new Approach to Generalized Associahedra* (2012), available at arXiv:1111.3349.

[91] Vincent Pilaud and Christian Stump, *EL-Labelings and Canonical Spanning Trees for Subword Complexes*, Fields Institute Communication, vol. 69, Springer, New York, 2013.

[92] Nathan Reading, *Lattice Congruences, Fans and Hopf Algebras*, Journal of Combinatorial Theory (Series A) **110** (2005), 237–273.

[93] Nathan Reading, *Cambrian Lattices*, Advances in Mathematics **205** (2006), 313–353.

[94] Nathan Reading, *Clusters, Coxeter-Sortable Elements and Noncrossing Partitions*, Transactions of the American Mathematical Society **359** (2007), 5931–5958.

[95] Nathan Reading, *Sortable Elements and Cambrian Lattices*, Algebra Universalis **56** (2007), 411–437.

[96] Nathan Reading, *Noncrossing Partitions and the Shard Intersection Order*, Journal of Algebraic Combinatorics **33** (2011), 483–530.

[97] Nathan Reading and David E. Speyer, *Sortable Elements in Infinite Coxeter Groups*, Transactions of the American Mathematical Society **363** (2011), 699–761.

[98] Victor Reiner, *Non-Crossing Partitions for Classical Reflection Groups*, Discrete Mathematics **177** (1997), 195–222.

[99] Victor Reiner, Vivien Ripoll, and Christian Stump, *On Non-Conjugate Coxeter Elements in Well-Generated Reflection Groups* (2014), available at arXiv:1404.5522.

[100] Vivien Ripoll, *Orbites d'Hurwitz des factorisations primitives d'un élément de Coxeter*, Journal of Algebra **323** (2010), 1432–1453.

[101] Gian-Carlo Rota, *On the Foundations of Combinatorial Theory I: Theory of Möbius Functions*, Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete **2** (1964), 340–368.

[102] Luigi Santocanale and Friedrich Wehrung, *Sublattices of Associahedra and Permutohedra*, Advances in Applied Mathematics **51** (2013), 419–445.

[103] Luis Serrano and Christian Stump, *Maximal Fillings of Moon Polyominoes, Simplicial Complexes, and Schubert Polynomials*, Electronic Journal of Combinatorics **19** (2012).

[104] Geoffrey C. Shephard and John A. Todd, *Finite Unitary Reflection Groups*, Canadian Journal of Mathematics **6** (1954), 274–304.

[105] Jian-yi Shi, *The Enumeration of Coxeter Elements*, Journal of Algebraic Combinatorics **6** (1997), 161–171.

[106] Rodica Simion, *Noncrossing Partitions*, Discrete Mathematics **217** (2000), 397–409.

[107] Rodica Simion, *A Type-B Associahedron*, Advances in Applied Mathematics **30** (2003), 2–25.

[108] Rodica Simion and Daniel Ullmann, *On the Structure of the Lattice of Noncrossing Partitions*, Discrete Mathematics **98** (1991), 193–206.

[109] Neil J. A. Sloane, *The Online Encyclopedia of Integer Sequences*. http://www.research.att.com/~njas/sequences/.

[110] Richard P. Stanley, *Catalan Addendum. New Problems for Enumerative Combinatorics, Vol. 2*. http://math.mit.edu/~rstan/ec/catadd.pdf.

[111] Richard P. Stanley, *Finite Lattices and Jordan-Hölder Sets*, Algebra Universalis **4** (1974), 361–371.

[112] Richard P. Stanley, *Enumerative Combinatorics, Vol. 1*, Cambridge University Press, Cambridge, 1997.

[113] Richard P. Stanley, *Enumerative Combinatorics, Vol. 2*, Cambridge University Press, Cambridge, 2001.

[114] James D. Stasheff, *Homotopy Associativity of H-Spaces. I*, Transactions of the American Mathematical Society **108** (1963), 275–292.

[115] James D. Stasheff, *Homotopy Associativity of H-Spaces. II*, Transactions of the American Mathematical Society **108** (1963), 293–312.

[116] James D. Stasheff, *From Operads to "Physically" Inspired Theories*, Contemporary Mathematics **202** (1997), 53–81.

[117] Robert Steinberg, *Differential Equations Invariant under Finite Reflection Groups*, Transactions of the American Mathematical Society **112** (1964), 392–400.

[118] Dov Tamari, *Monoides Préordonnés et Chaînes de Malcev*, Thèse de Mathématiques, Université de Paris, 1951.

[119] Dov Tamari, *The Algebra of Bracketings and their Enumeration*, Nieuw Archief voor Wiskunde **10** (1962), 131–146.

[120] Dov Tamari, *Sur Quelques Problèmes d'Associativité*, Annales Scientifiques de l'Université de Clermont-Ferrand 2, Série Mathématiques **24** (1964), 91–107.

[121] Hugh Thomas, *An Analogue of Distributivity for Ungraded Lattices*, Order **23** (2006), 249–269.

[122] Hugh Thomas, *Tamari Lattices and Noncrossing Partitions in Type B*, Discrete Mathematics **306** (2006), 2711–2723.

[123] Masaya Tomie, *Möbius Numbers of Some Modified Generalized Noncrossing Partitions* (2009), available at arXiv:0905.1660v2.

[124] Eleni Tzanaki, *Polygon Dissections and some Generalizations of Cluster Complexes*, Journal of Combinatorial Theory (Series A) **113** (2006), 1189–1198.

[125] Alasdair Urquhart, *A Topological Representation Theory for Lattices*, Algebra Universalis **8** (1978), 45–58.

[126] Nikolaus von Fuß, *Solutio Quaestionis quot Modis Polygonum n Laterum in Polygona m Laterum per Diagonales Resolvi queat*, Nova Acta Academiae Scientiarum Imperialis Petropolitanae **IX** (1791), 243–251.

[127] Michelle L. Wachs, *Poset Topology: Tools and Applications*, IAS/Park City Mathematics Series, vol. 13, American Mathematical Society, Providence, RI, 2007.

[128] Hermann Weyl, *Theorie der Darstellung kontinuierlicher halbeinfacher Gruppen durch lineare Transformationen I*, Mathematische Zeitschrift **23** (1925), 271–309.

[129] Hermann Weyl, *Theorie der Darstellung kontinuierlicher halbeinfacher Gruppen durch lineare Transformationen II*, Mathematische Zeitschrift **24** (1926), 328–395.

[130] Hermann Weyl, *Theorie der Darstellung kontinuierlicher halbeinfacher Gruppen durch lineare Transformationen III*, Mathematische Zeitschrift **24** (1926), 789–791.

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Research Interests

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Fuß-Catalan combinatorics.
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Awards

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Publications and Preprints

11. *Counting Proper Mergings of Chains and Antichains*. Discrete Mathematics 327(C). Elsevier, 2014.

10. *Structural Properties of the Cambrian Semilattices – Consequences of Semidistributivity*. Preprint, arXiv:1312.4449, 2013.

9. *The m-Cover Posets and the Strip-Decomposition of m-Dyck Paths* (with M. Kallipoliti). Preprint, arXiv:1312.2520, 2013.

8. *A Heyting Algebra on Dyck Paths of Type A and B*. Preprint, arXiv:1312.0551, 2013.

7. *Proper Mergings of Stars and Chains are Counted by Sums of Antidiagonals in Certain Convolution Arrays*. Proceedings of the 11th International Conference on Formal Concept Analysis, Dresden, Germany. Springer, 2013.

6. *On the Topology of the Cambrian Semilattices* (with M. Kallipoliti). The Electronic Journal of Combinatorics 20(2), 2013.
   (Extended Abstract published in Proceedings of the 25th International Conference on Formal Power Series and Algebraic Combinatorics, Paris, France. DMTCS, 2013.)

5. *On the EL-Shellability of the m-Tamari Lattices*. Preprint, arXiv:1201.2020, 2012.

4. *EL-Shellability of Generalized Noncrossing Partitions Associated with Well-Generated Complex Reflection Groups*. Preprint, arXiv:1111.7172, 2011.
   (Extended Abstract published in Proceedings of the 24th International Conference on Formal Power Series and Algebraic Combinatorics, Nagoya, Japan. DMTCS, 2012.)

3. *Merging Ordered Sets* (with B. Ganter and C. Meschke). Proceedings of the 9th International Conference on Formal Concept Analysis, Nicosia, Cyprus. Springer, 2011.

2. *Using Bonds for Describing Method Dispatch in Role-Oriented Software Models*. Proceedings of the 7th International Conference on Concept Lattices and their Applications, Sevilla, Spain. University of Sevilla, 2010.

1. *Describing Role-Oriented Software Models in Terms of Formal Concept Analysis* (with C. Wende). Proceedings of the 8th International Conference on Formal Concept Analysis, Agadir, Morocco. Springer, 2010.

Talks and Posters

10. *On m-Cover Posets and Their Applications* at the 72nd Séminaire Lotharingien de Combinatoire in Lyon, France, 2014. (Talk)
9. *On the Topology of the Cambrian Semilattices* at CoMeTA'13 in Cortona, Italy, 2013. (Poster)
8. *On the Topology of the Cambrian Semilattices* at the 25th International Conference on Formal Power Series and Algebraic Combinatorics in Paris, France, 2013. (Poster)
7. *Proper Mergings of Stars and Chains are Counted by Sums of Antidiagonals in Certain Convolution Arrays* at the 11th International Conference on Formal Concept Analysis in Dresden, Germany, 2013. (Talk)
6. *Counting Proper Mergings* at the 70th Séminaire Lotharingien de Combinatoire in Ellwangen, Germany, 2013. (Talk)
5. *The Cambrian Lattices are EL-Shellable* at the 69th Séminaire Lotharingien de Combinatoire in Strobl, Austria, 2012. (Talk)
4. *EL-Shellability of Generalized Noncrossing Partitions Associated to Well-Generated Complex Reflection Groups* at the 24th International Conference on Formal Power Series and Algebraic Combinatorics in Nagoya, Japan, 2012. (Poster)
3. *EL-Shellability of the m-Tamari Lattices* at the 68th Séminaire Lotharingien de Combinatoire in Ottrott, France, 2012. (Talk)
2. *Using Bonds for Describing Method Dispatch in Role-Oriented Software Models* at the 7th International Conference on Concept Lattices and their Applications in Seville, Spain, 2010. (Talk)
1. *Describing Role-Oriented Software Models in Terms of Formal Concept Analysis* at the 8th International Conference on Formal Concept Analysis in Agadir, Morocco, 2010. (Talk)